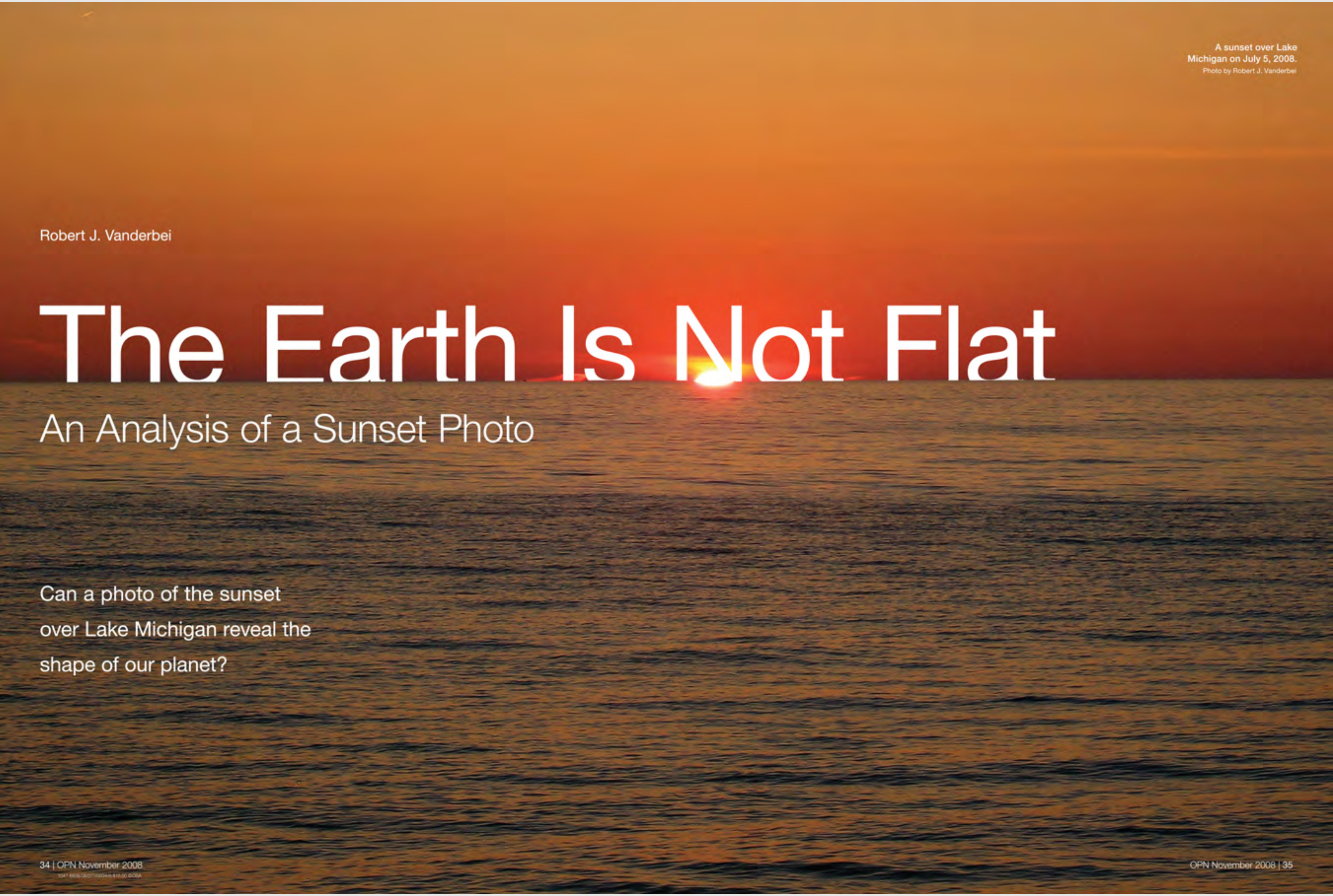


A sunset over Lake Michigan on July 5, 2008.
Photo by Robert J. Vanderbei

Robert J. Vanderbei

The Earth Is Not Flat

An Analysis of a Sunset Photo

Can a photo of the sunset over Lake Michigan reveal the shape of our planet?

I will show you how we can...

measure something *BIG* (the size of the Earth)

by first measuring something *small* (my height),
and measuring an *angle* (off from a photograph)

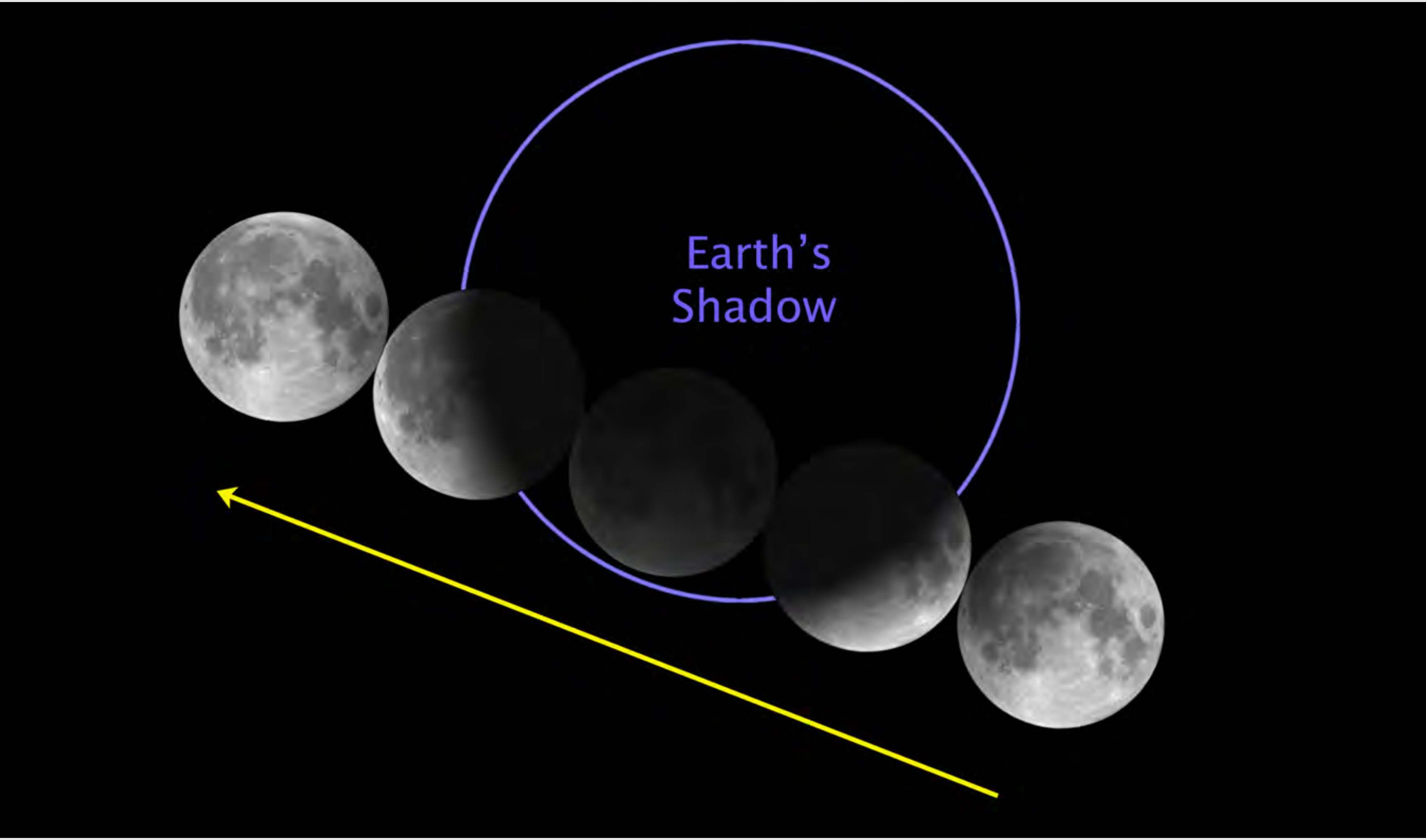
and then doing some *geometry*.

The Earth is a big sphere. How do we know?

Several ways. One way is to look at a Lunar eclipse...

Photo taken March 3, 2007, at about 8pm.

From a lunar eclipse, we can determine that the Earth is about 3 or 4 times larger than the Moon. But, how big is the Earth?



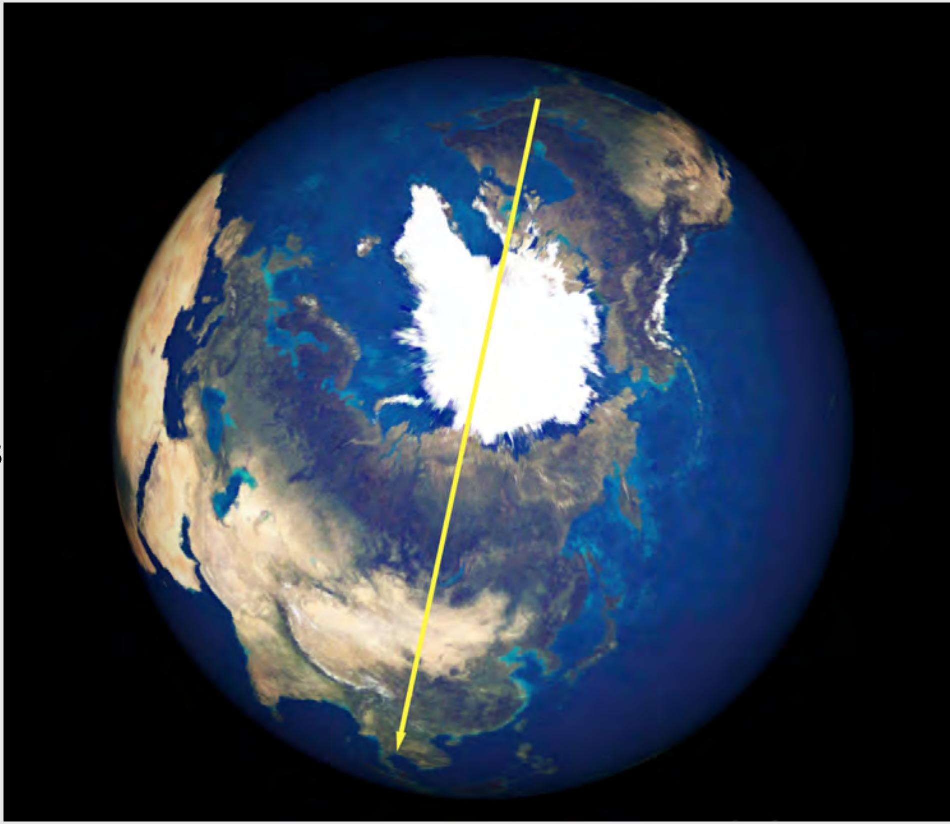
Next total lunar eclipse visible from the "east coast" is on January 20/21, 2019.

How big is the Earth? How can we find out?

First Method: Look it up on Wikipedia.

You'll get the right answer (radius $= 3{,}960$ miles), but no satisfaction.

Second Method: Air travel.



I've flown to Bangkok Thailand.
It's about a 17 hour flight.
It's about halfway around the Earth.
Jets fly at about 600 mph.
So, the distance I flew is *about*

17 hours$\times 600$ miles/hour $= 10{,}200$ miles

The circumference is then about $20{,}000$ miles and radius is therefore about $20{,}000/2\pi = 3{,}250$ miles. This is just a rough estimate.

IS THERE AN EASIER WAY?

A picture I took of a sunset over Lake Michigan.

A close-up.

Using this picture, some geometry, and a little trigonometry, I was able to compute that the Earth's radius is about 5000 miles.

A smooth lake is supposed to act like a mirror.

Credit: Lorene Lavora

The Sun's reflection should have looked something like this...

Or not!

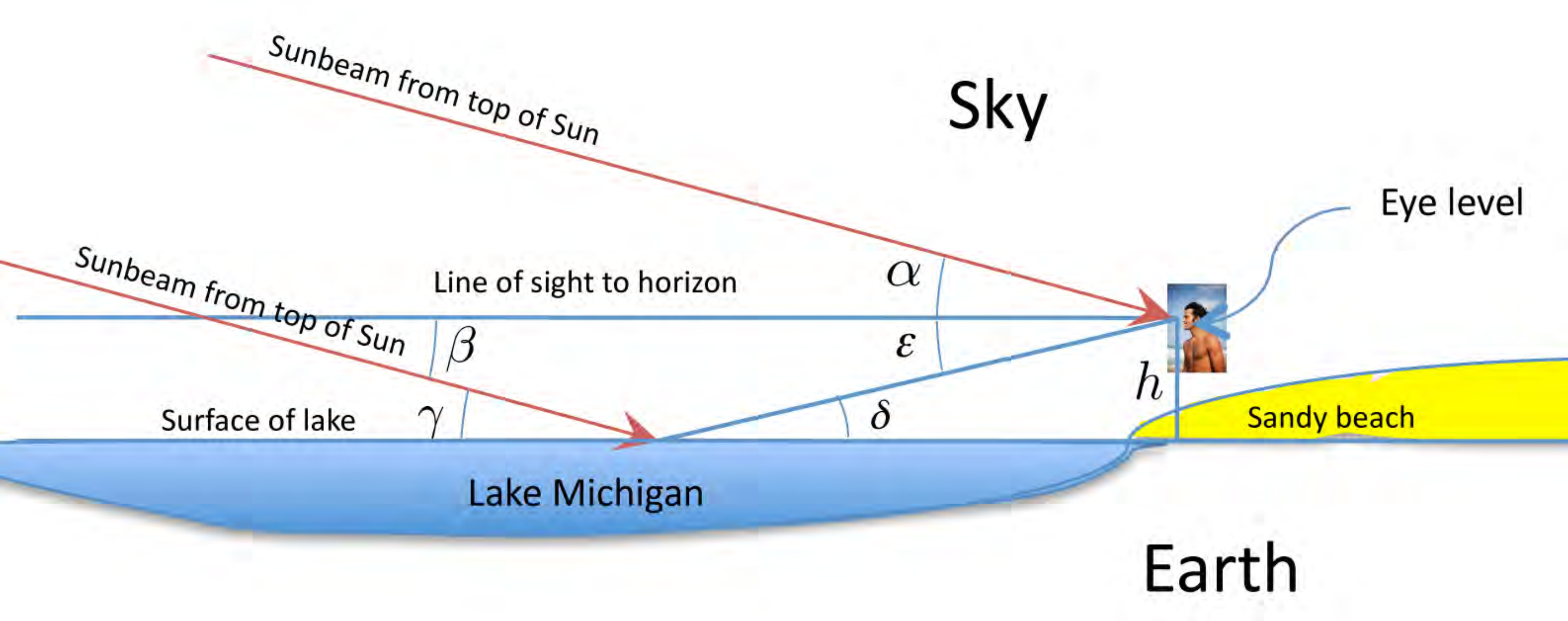
What's going on?

Lake Michigan is not a flat mirror.

Its surface is curved because the Earth is a sphere.

That's why we can't see the shore on the opposite side—it's below the horizon!

Geometry — If the Earth Were Flat!



$\alpha = \beta$		alternate interior angles are equal
$\beta = \gamma$		alternate interior angles are equal
$\gamma = \delta$		angle of incidence equals angle of reflection (from Physics!)
$\delta = \epsilon$		alternate interior angles are equal

Therefore,

$$\alpha = \epsilon.$$

The reflection dips just as far below the horizon as the Sun stands above the horizon.

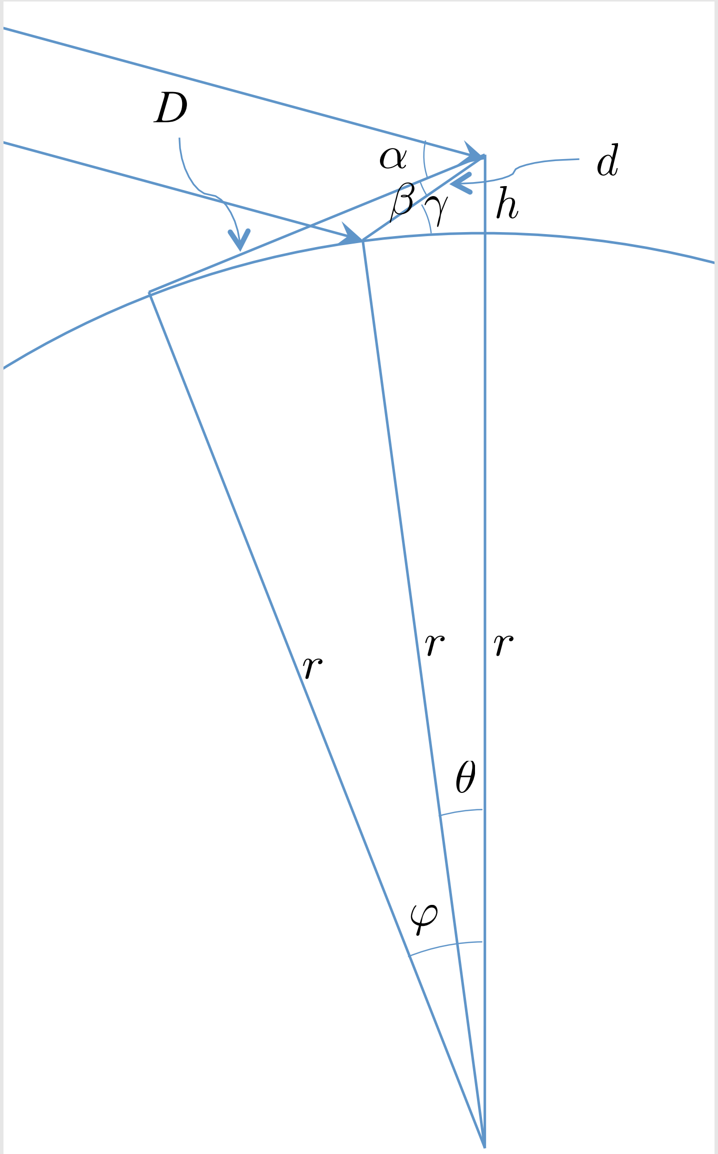
Geometry — The Earth Is Not Flat

Draw a picture.

Label everything of possible relevance.

Identify what we know:

- α Angle between horizon and top of Sun (measured from photo)
- β Angle between horizon and "top" of Sun in reflection (measured)
- h Height of "eye-level" above "water-level".



Geometry — The Earth Is Not Flat

Everything else is the stuff we need to figure out:

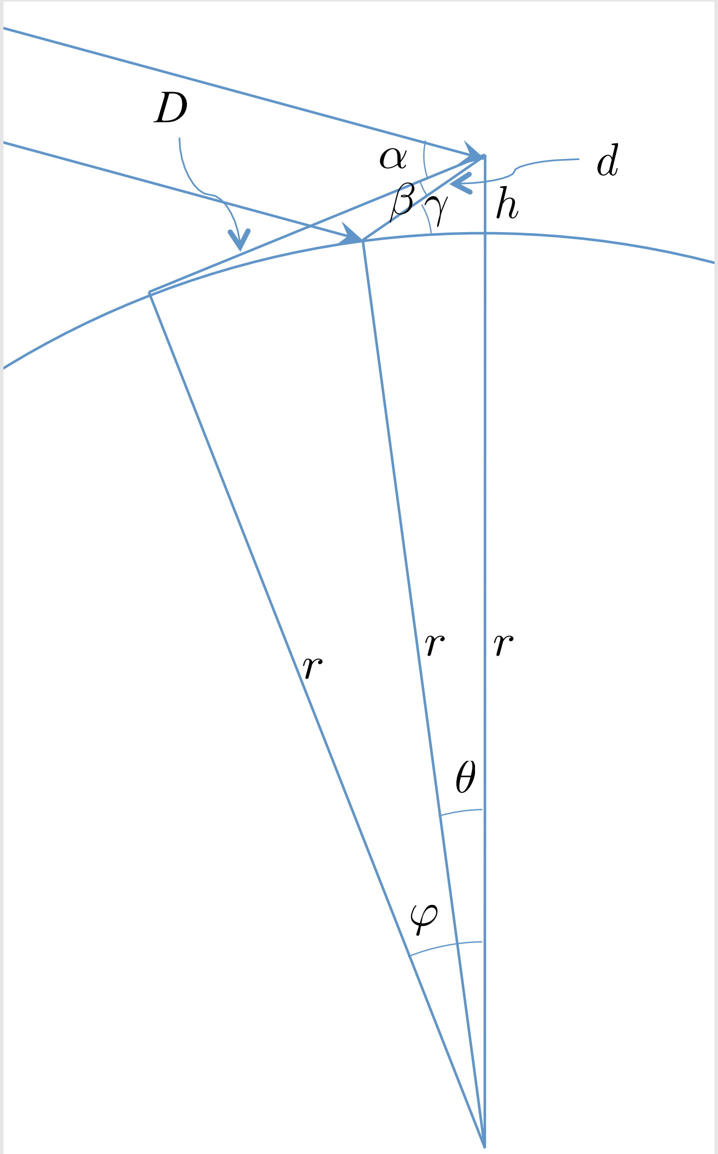
- Three Angles:

  γ Angle of reflection off water.

  θ Angle between observer (me) and point of reflection.

  φ Angle between observer (me) and point of horizon.

Plus...



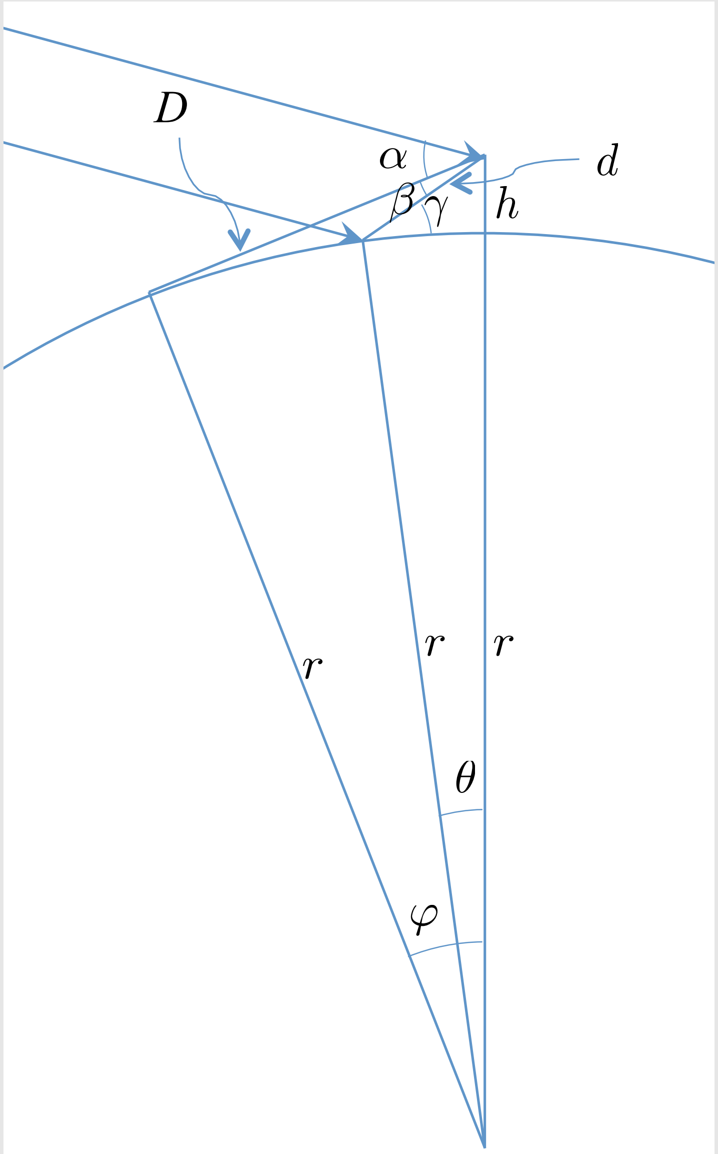
Geometry — The Earth Is Not Flat

- Three Distances (lengths):

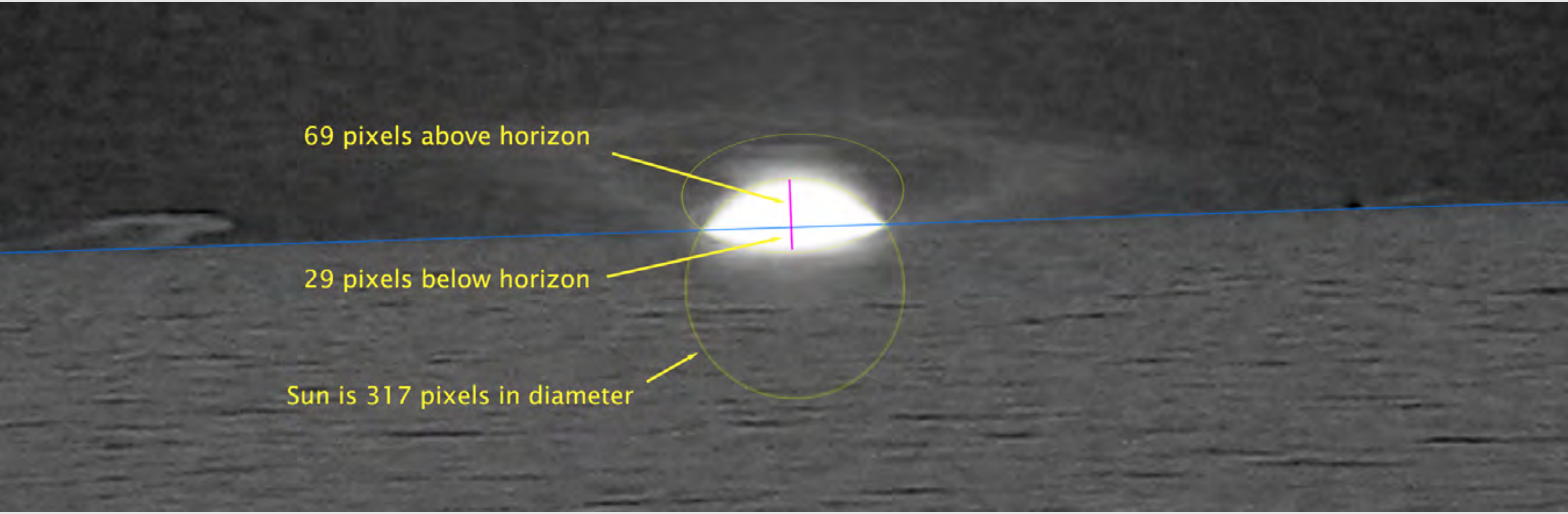
  d Distance to point of reflection.

  D Distance to horizon.

  r Radius of Earth. $\Longleftarrow$ This one is key!!!



What We Know (Measure!)



The Sun is $1/2^\circ$ in diameter. Therefore, 1° equals $2 \times 317 = 634$ pixels.

And so,

$$\alpha = 69 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.1088 \text{ degrees}$$

and

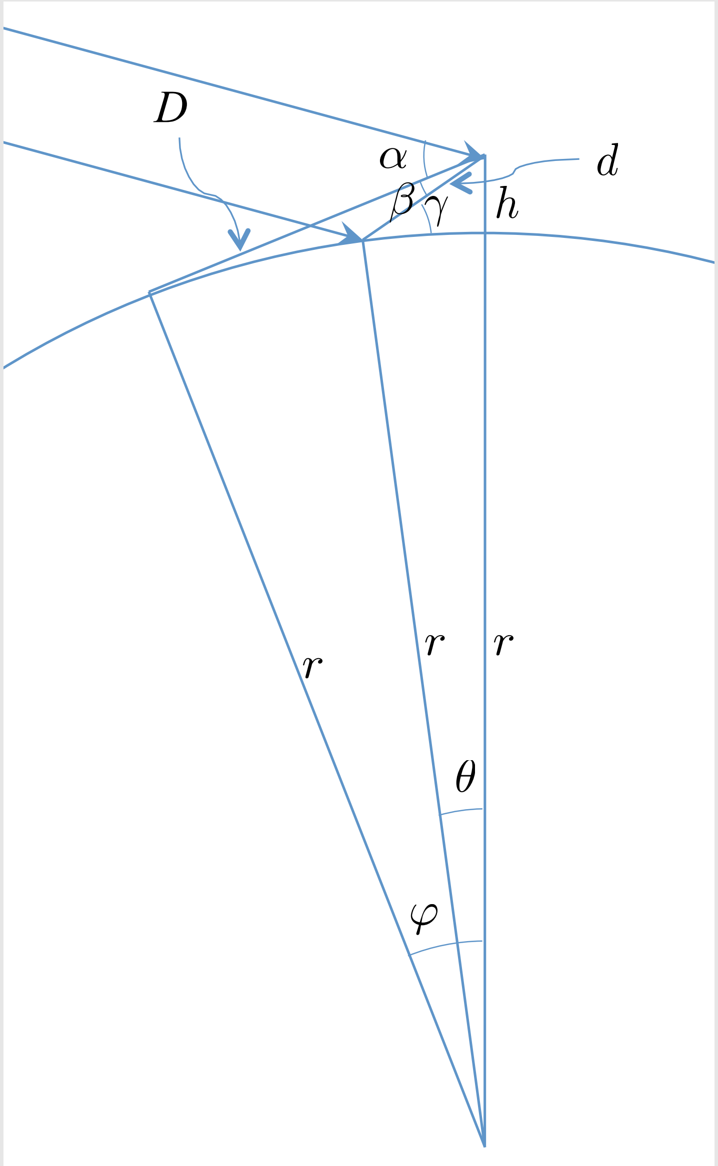
$$\beta = 29 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.0457 \text{ degrees.}$$

And, we assume that eye level is

$$h = 7 \text{ feet}$$

What We Need To Figure Out:

- Angles:
  - γ Angle of reflection off water.
  - θ Angle between observer (me) and point of reflection.
  - φ Angle between observer (me) and point of horizon.
- Distances (lengths):
  - d Distance to point of reflection.
  - D Distance to horizon.
  - r Radius of Earth. $\Longleftarrow$ This one is key!!!

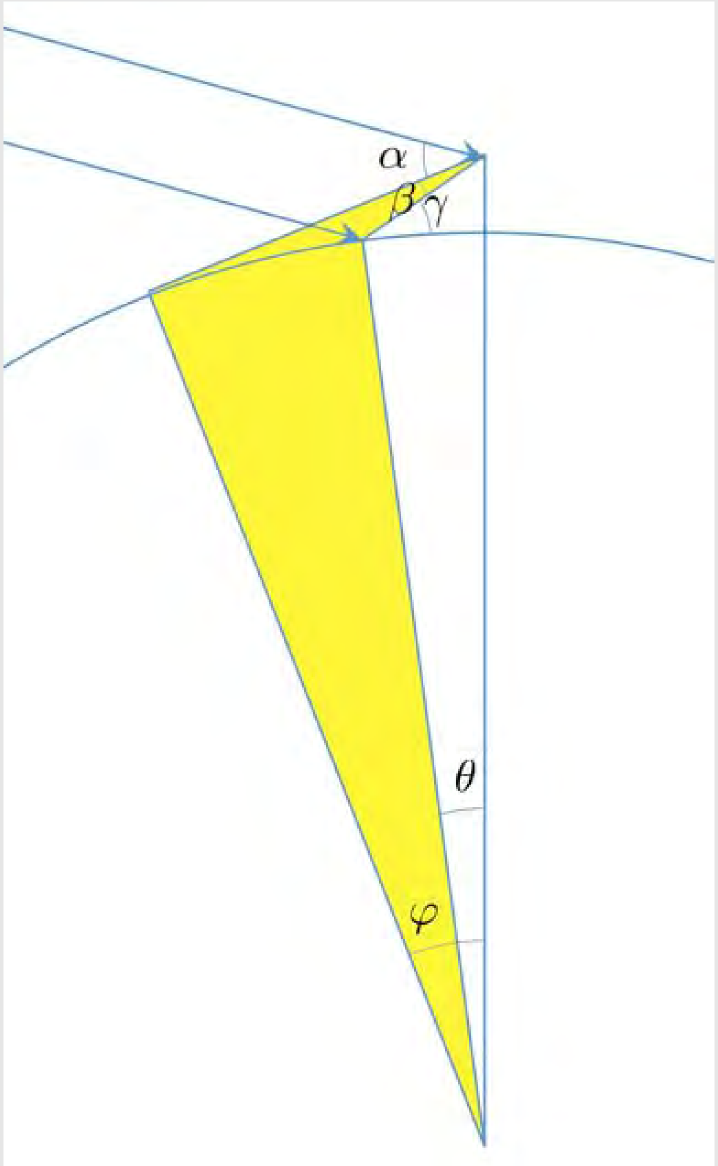


That's SIX UNKNOWNS.

We need SIX (distinct!) EQUATIONS.

Equation 1:

The sum of the angles around a quadrilateral is $360°$.



Hence,

$$(\varphi - \theta) + 90 + \beta + (270 - \gamma) = 360.$$

Simplifying, we get

$$\varphi + \beta = \theta + \gamma.$$

Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to 180°.

Hence,

$$\alpha + \beta + \sigma = 180.$$

Also, because angle of incidence equals angle of reflection, we see that

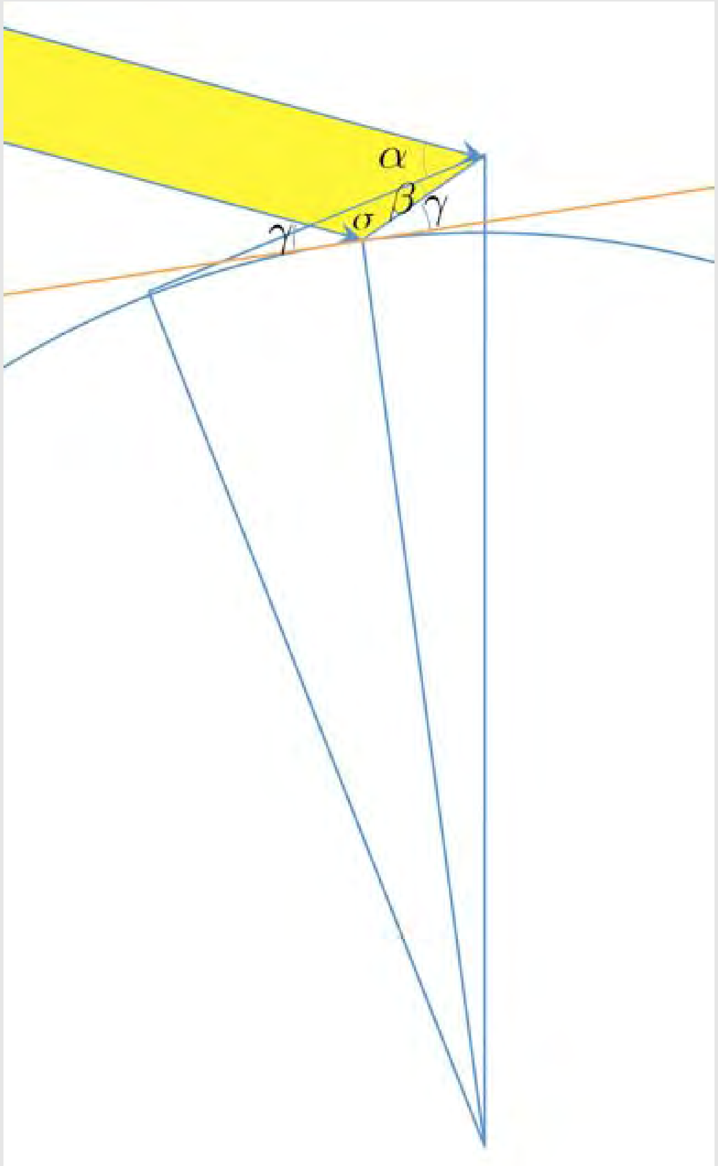
$$\gamma + \sigma + \gamma = 180.$$

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$

So, our second equation is

$$\alpha + \beta = 2\gamma.$$



Equation 3:

The distance from the center of the Earth to eye level is

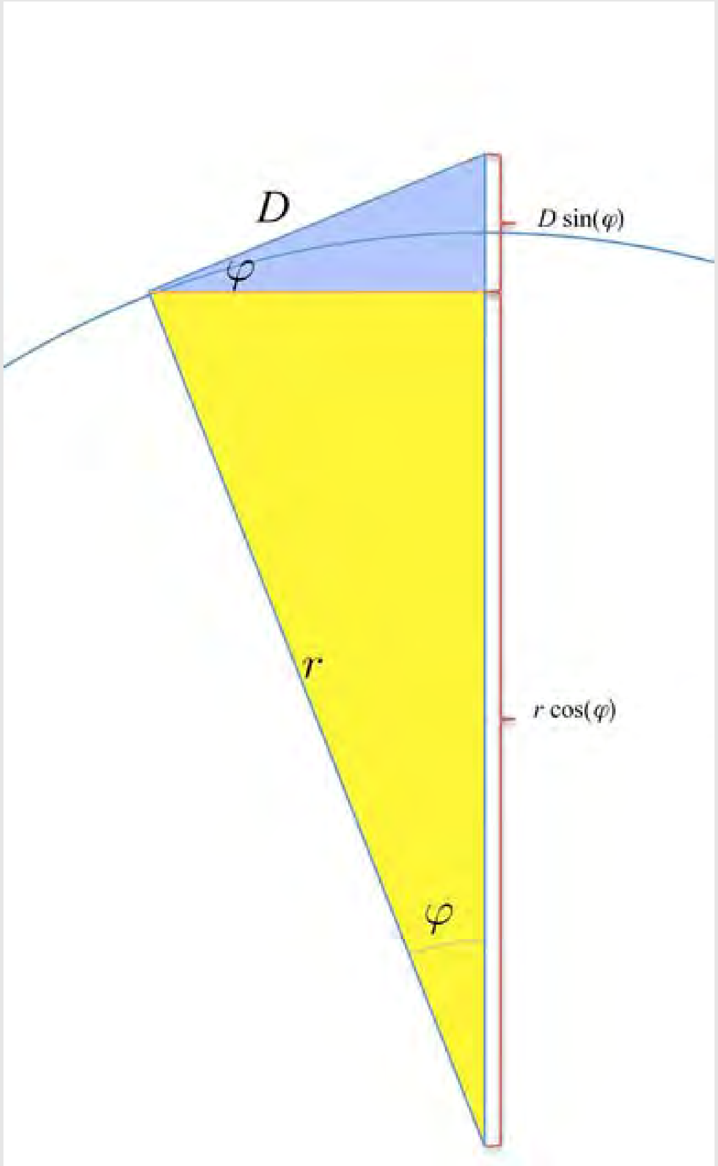
$$r + h.$$

But, it is also

$$D\sin(\varphi) + r\cos(\varphi).$$

Hence,

$$D\sin(\varphi) + r\cos(\varphi) = r + h.$$



Equation 4:

The “horizontal” distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

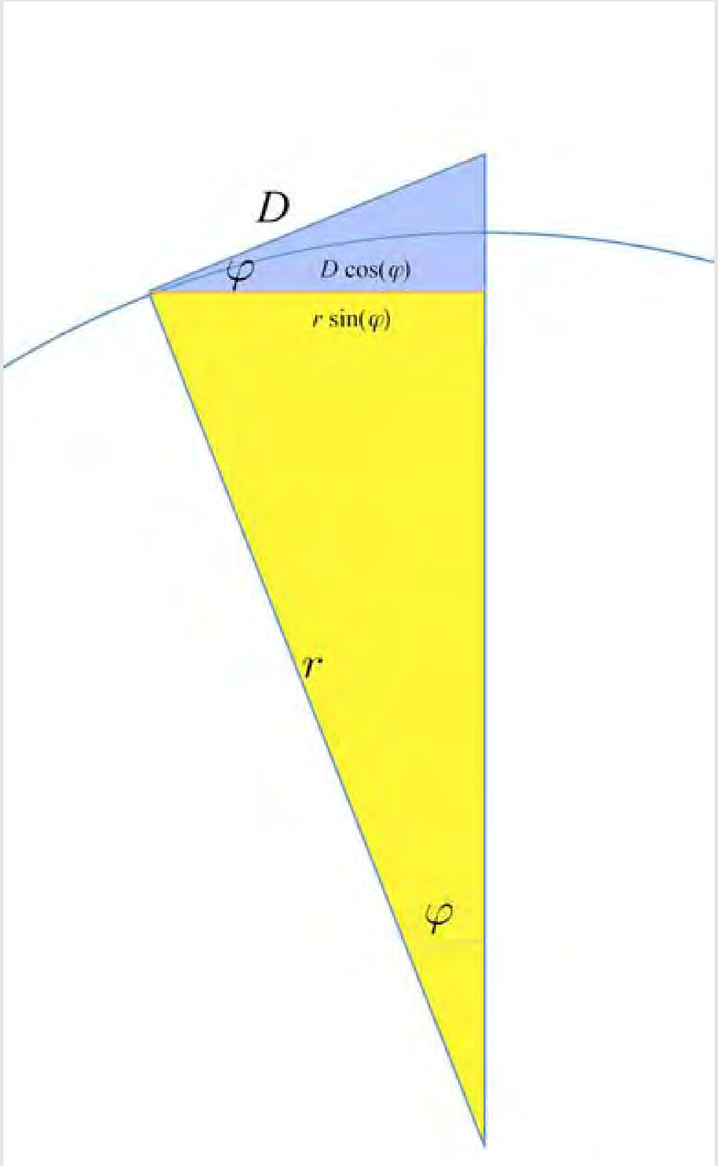
$$D\cos(\varphi)$$

and

$$r\sin(\varphi).$$

Hence,

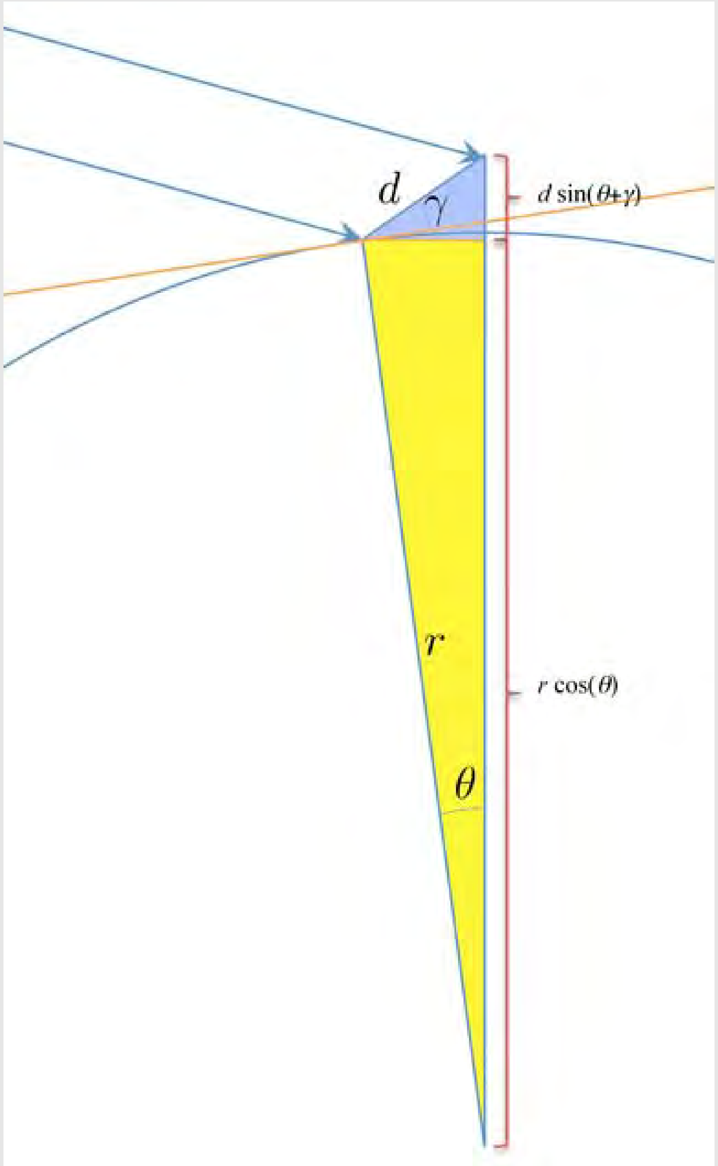
$$D\cos(\varphi) = r\sin(\varphi).$$



Equation 5:

Equation 5 is analogous to Equation 3, using the "point of reflection" in place of the "horizon".

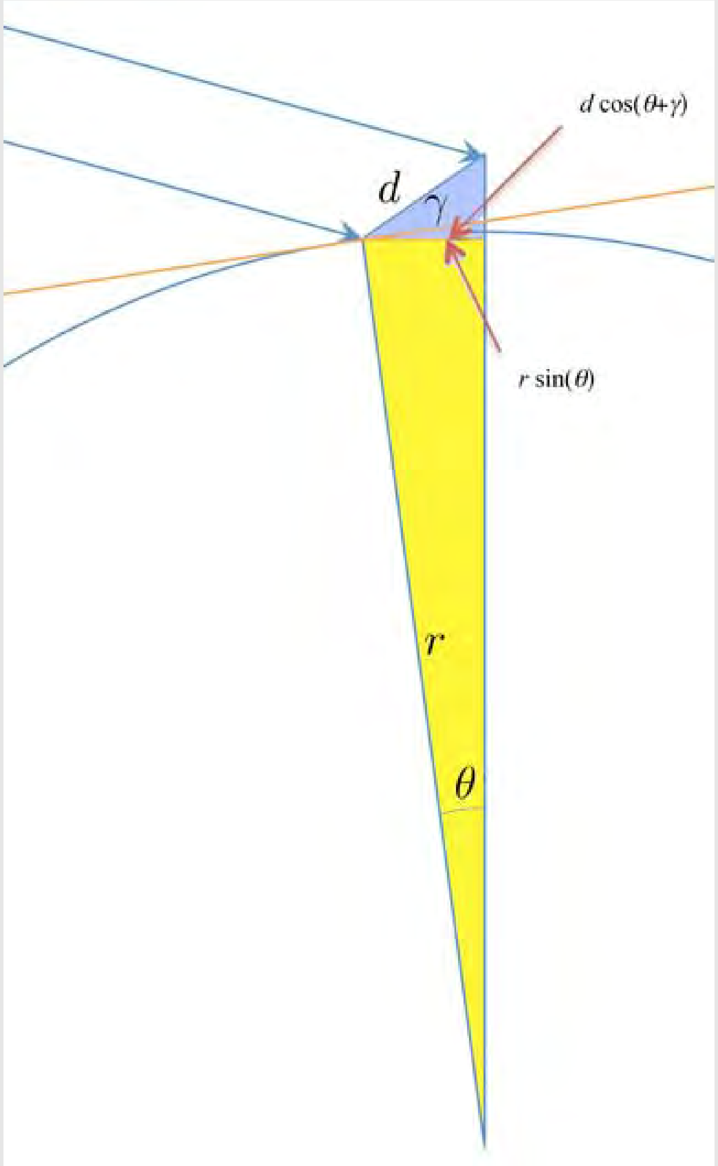
$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h.$$



Equation 6:

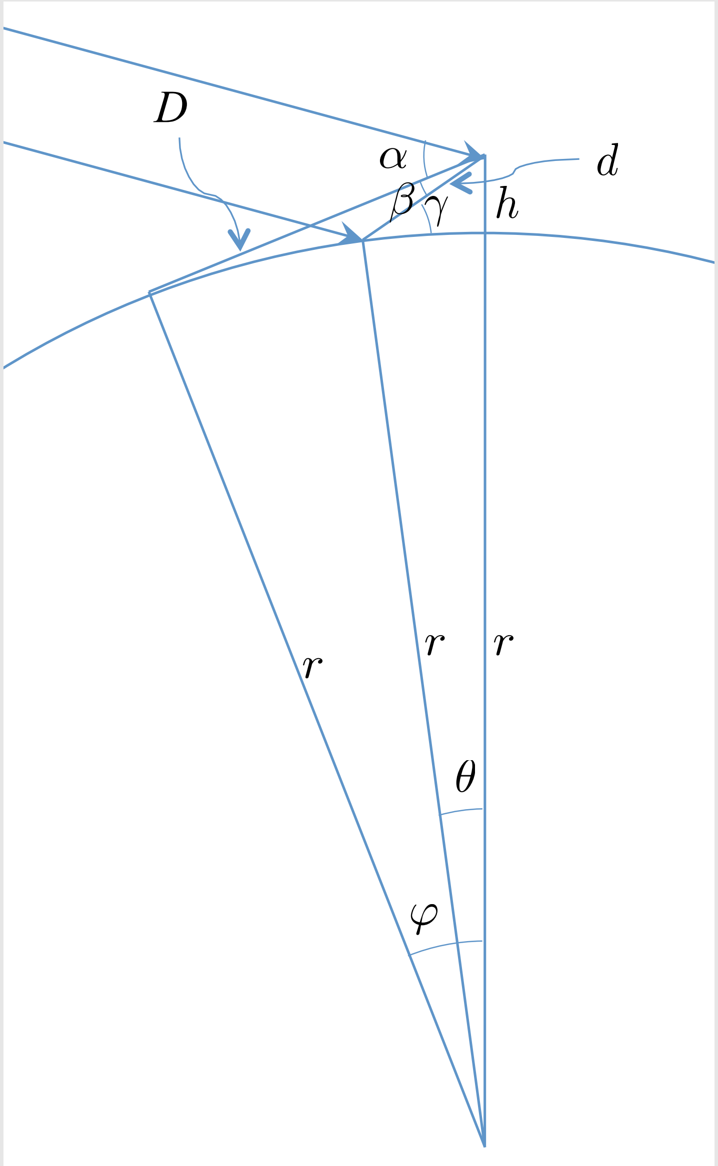
Equation 6 is analogous to Equation 4 in the same way.

$$d\cos(\theta + \gamma) = r\sin(\theta).$$



Six Equations in Six Unknowns:

$$
\begin{aligned}
\varphi + \beta &= \theta + \gamma && (1)\\
\alpha + \beta &= 2\gamma && (2)\\
D\sin(\varphi) + r\cos(\varphi) &= r + h && (3)\\
D\cos(\varphi) &= r\sin(\varphi) && (4)\\
d\sin(\theta+\gamma) + r\cos(\theta) &= r + h && (5)\\
d\cos(\theta+\gamma) &= r\sin(\theta). && (6)
\end{aligned}
$$



Not hard to solve.

Use (2) to solve for γ.
Solve (4) for D and then substitute in for D in (3).
Solve (6) for d and then substitute in for d in (5).
And so on...

Three Equations in Three Unknowns:

$$\gamma = (\alpha + \beta)/2 \quad (2)$$
$$D = r \sin(\varphi)/\cos(\varphi) \quad (4)$$
$$d = r \sin(\theta)/\cos(\theta + \gamma) \quad (6)$$

$$\varphi - \theta = (\alpha - \beta)/2 \quad (1)$$
$$r = (r + h)\cos(\varphi) \quad (3)$$
$$r\cos(\gamma) = (r + h)\cos(\theta + \gamma) \quad (5)$$

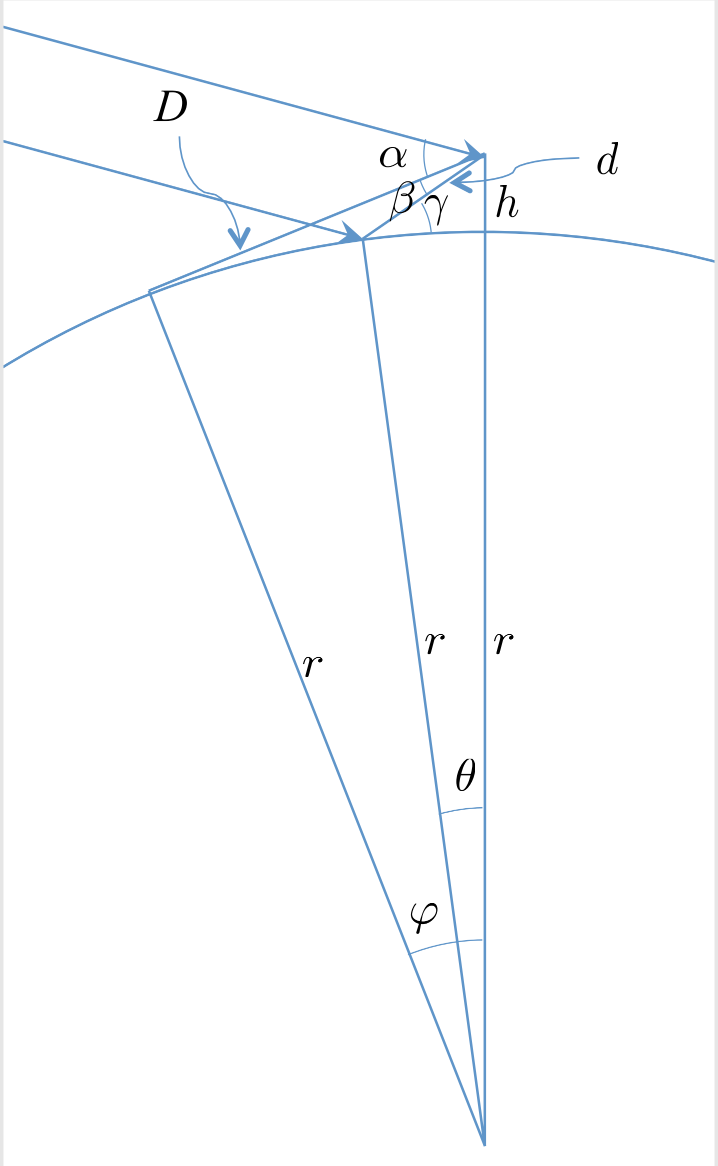
Divide (3) and (5) by $r + h$ and eliminate r:

$$\cos(\varphi) = \cos(\theta + \gamma)/\cos(\gamma) = \cos(\varphi + \beta)/\cos(\gamma)$$

Expand the cosine of the sum, replace $\sin(\varphi)$ with $\sqrt{1 - \cos^2(\varphi)}$ and solve for $\cos(\varphi)$:

$$\cos(\varphi) = \frac{\sin(\beta)}{\sqrt{1 - 2\cos(\beta)\cos(\gamma) + \cos^2(\gamma)}}$$

Substitute this formula for $\cos(\varphi)$ into (3) and solve for r...



Answer for r (radius of Earth) is:

$$r = \frac{h}{\frac{\sqrt{1 - 2\cos\beta\cos\gamma + \cos^2\gamma}}{\sin\beta} - 1}$$

where

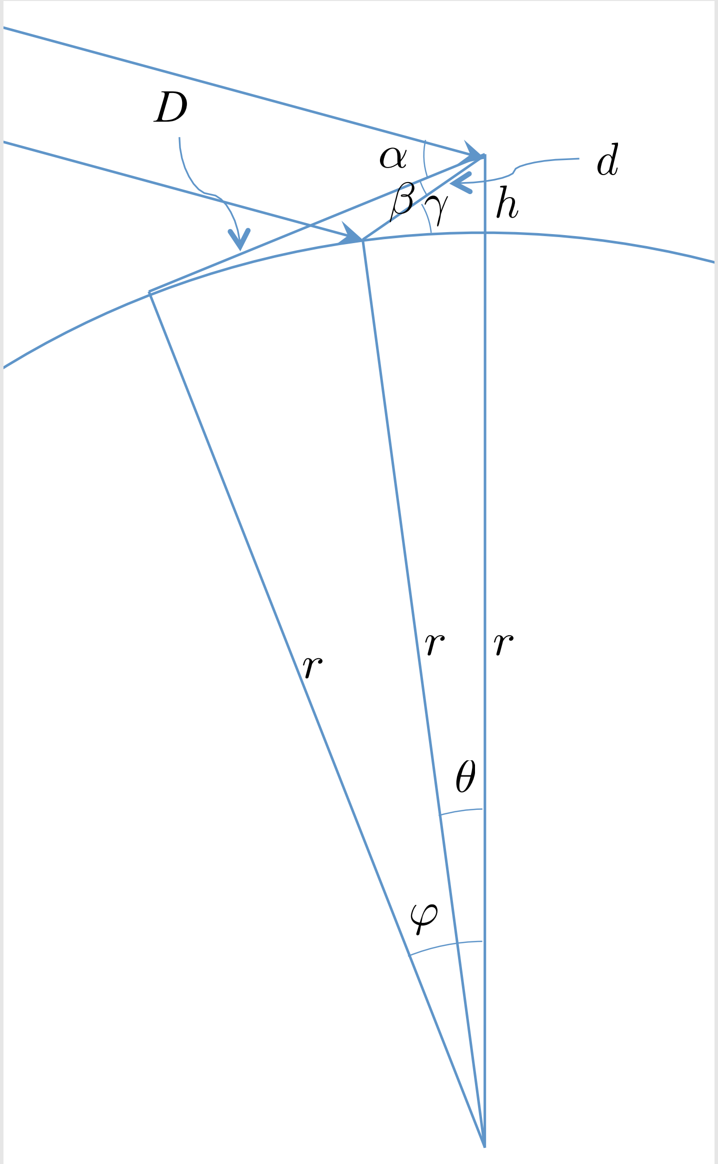
$$\gamma = \frac{\alpha + \beta}{2}.$$

Plugging in our values for α, β, and h, we get

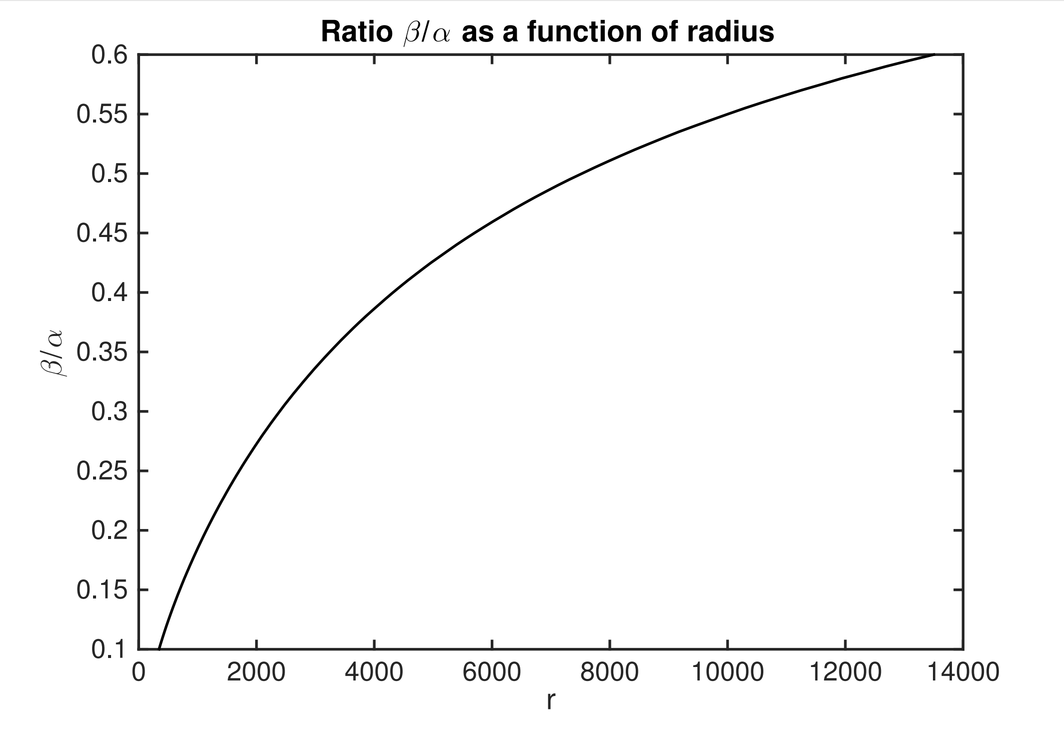
$$r = 4{,}977 \text{ miles.}$$

Recall that the right answer is 3,960 miles.
Fixing α and β, the height h that corresponds to this answer is:

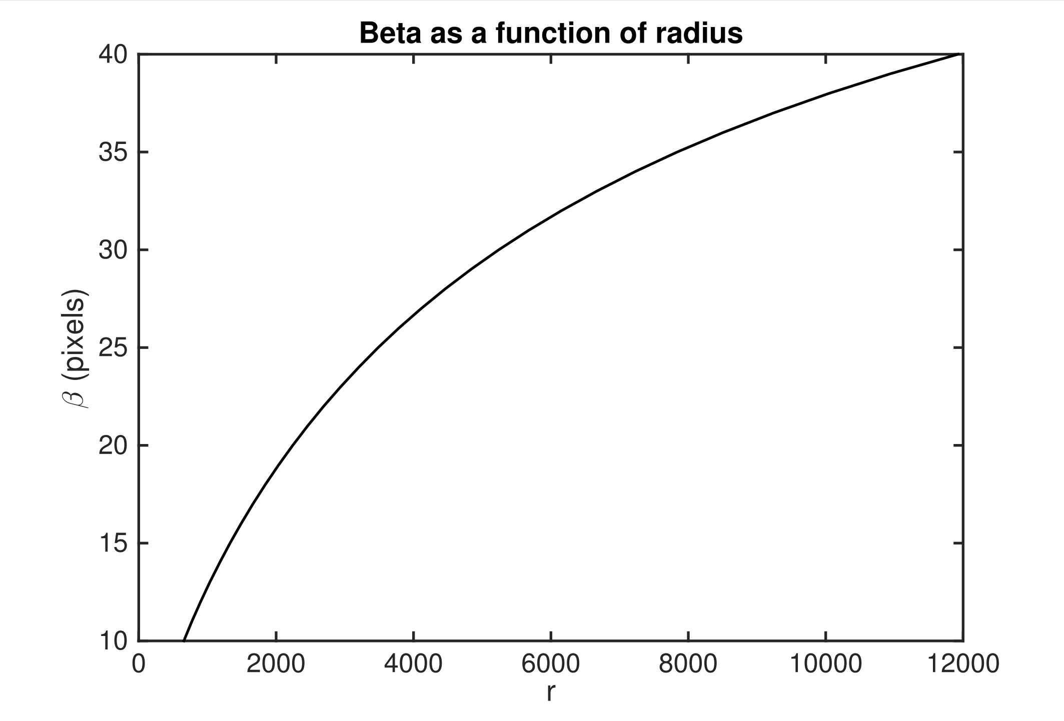
$$h = 7 \times \frac{3960}{4977} = 5.56 \text{ feet} = 5' \, 7''.$$



Fix α. How does the ratio β/α vary with r...



In terms of pixels...



Morals:

- Always be mindful of units.
- Always draw a picture and label things.
- If there are six unknowns, you need six (distinct) equations.
- A picture need not be to scale; it can exagerate angles, distances, etc.
-
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Conclusion: ALGEBRA AND GEOMETRY ARE BOTH FUN AND USEFUL.