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# The Earth Is Not Flat

An Analysis of a Sunset Photo

Can a photo of the sunset  
over Lake Michigan reveal the  
shape of our planet?

I will show you how we can...

measure something *BIG* (the size of the Earth)

by first measuring something *small* (my height),  
measuring an *angle* (off from a photograph)

and then doing some *geometry*.

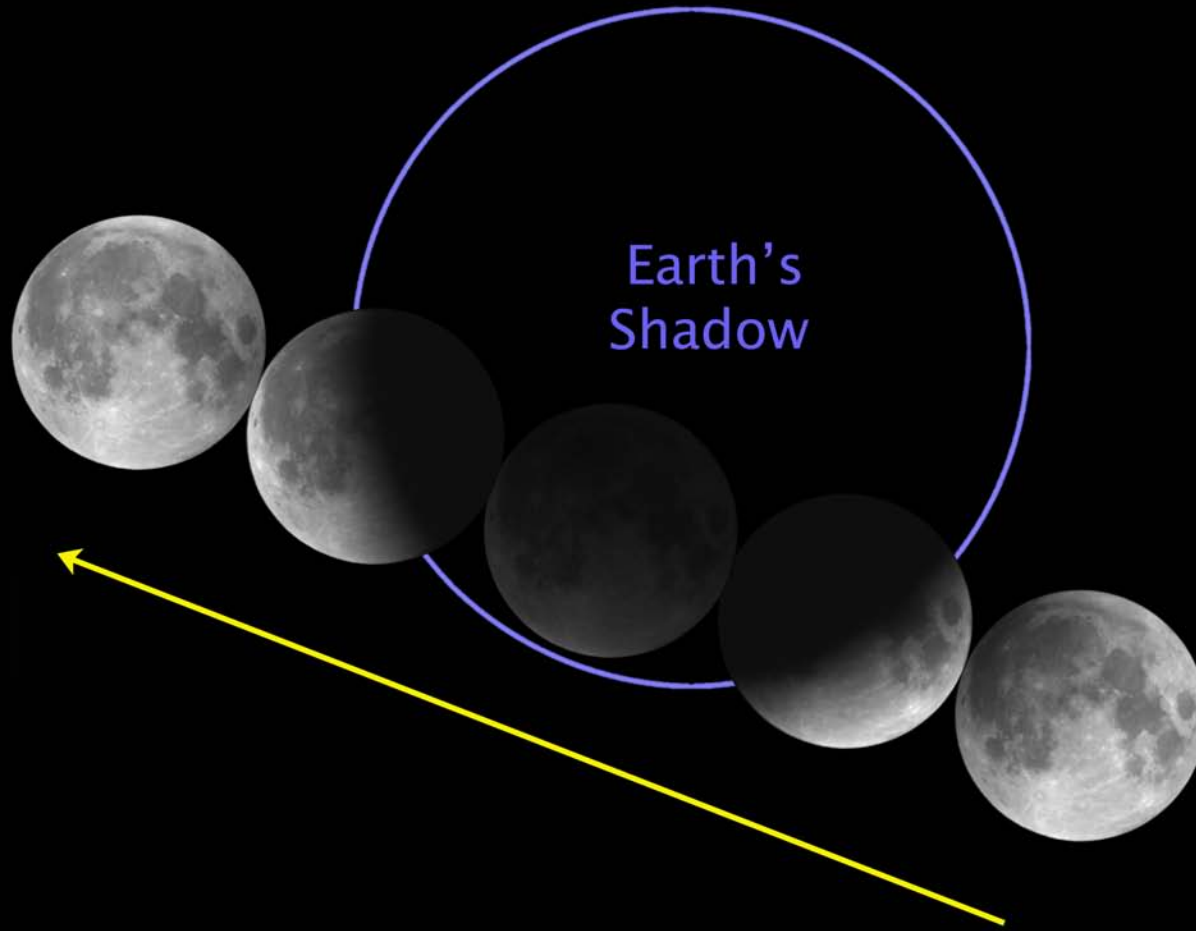
The Earth is a big sphere. How do we know?

Several ways. One way is to look at a Lunar eclipse...



Photo taken March 3, 2007, at about 8pm.

From a Lunar eclipse, we can determine that the Earth is about 3 or 4 times larger than the Moon. But, how big is the Earth?



Next Lunar eclipse is late at night on December 20/21 (starts after midnight).

How big is the Earth? How can we find out?

First Method: Look it up on Wikipedia.

You'll get the right answer (radius = 3,960 miles), but no satisfaction.

Second Method: Air travel.

I've flown to Bangkok Thailand.

It's about a 17 hour flight.

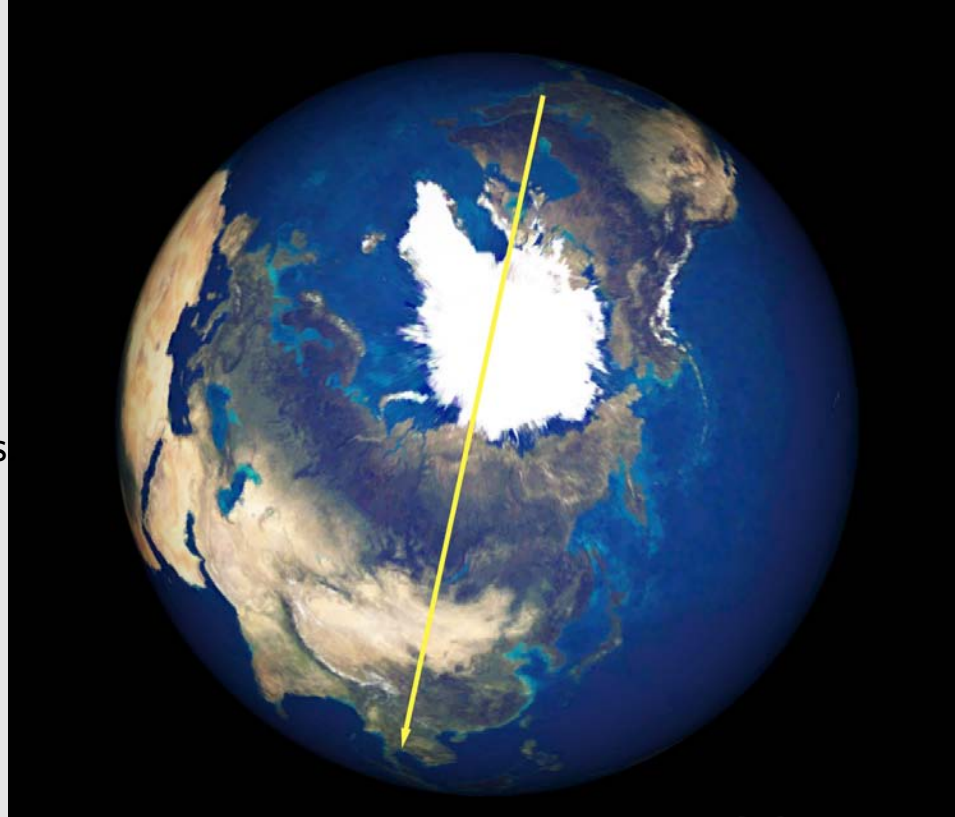
It's about halfway around the Earth.

Jets fly at about 600 mph.

So, the distance I flew is *about*

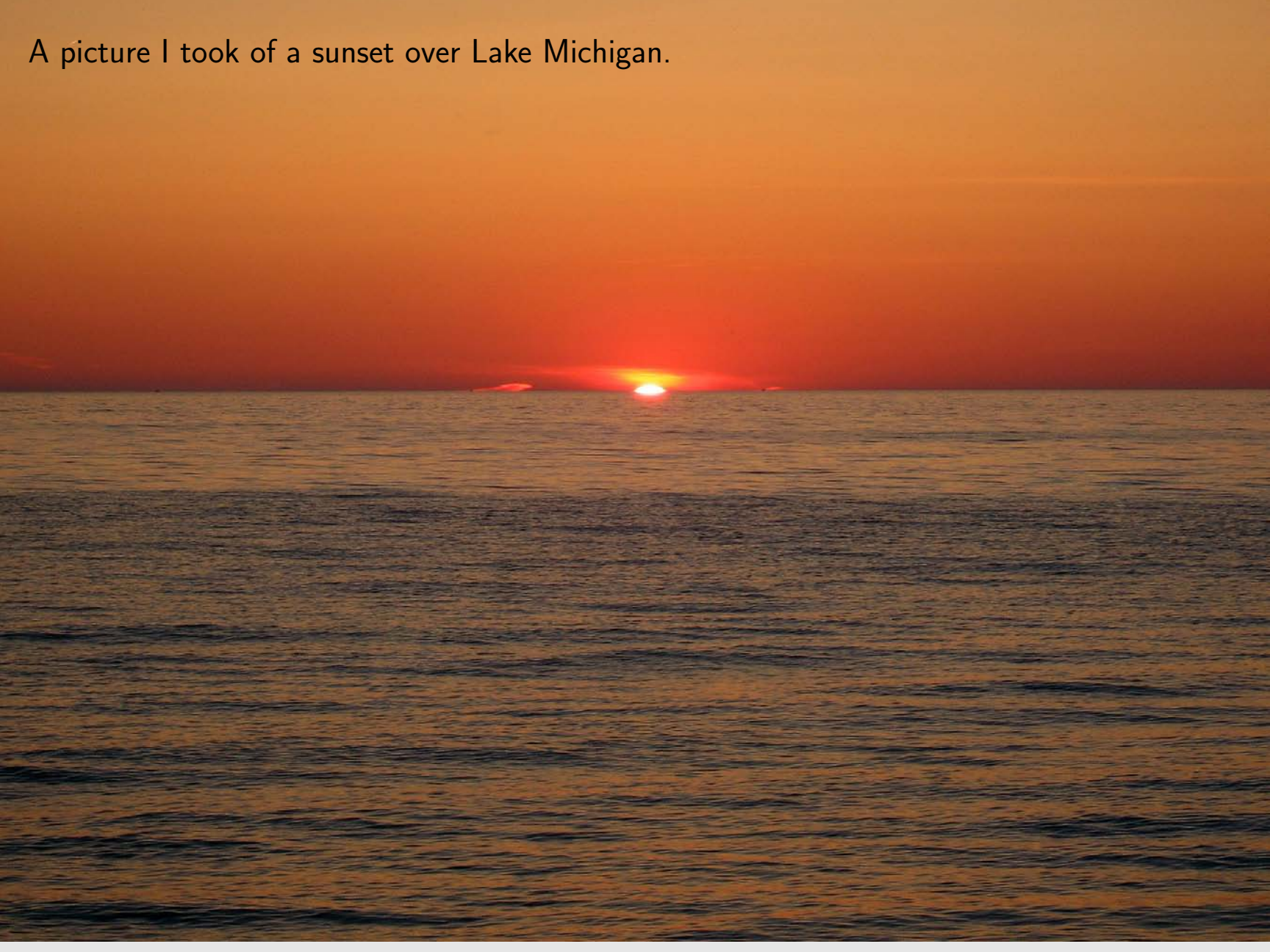
$$17 \text{ hours} \times 600 \text{ miles/hour} = 10,200 \text{ miles}$$

The circumference is then about 20,000 miles and radius is therefore about  $20,000/2\pi = 3,250$  miles. This is just a rough estimate.



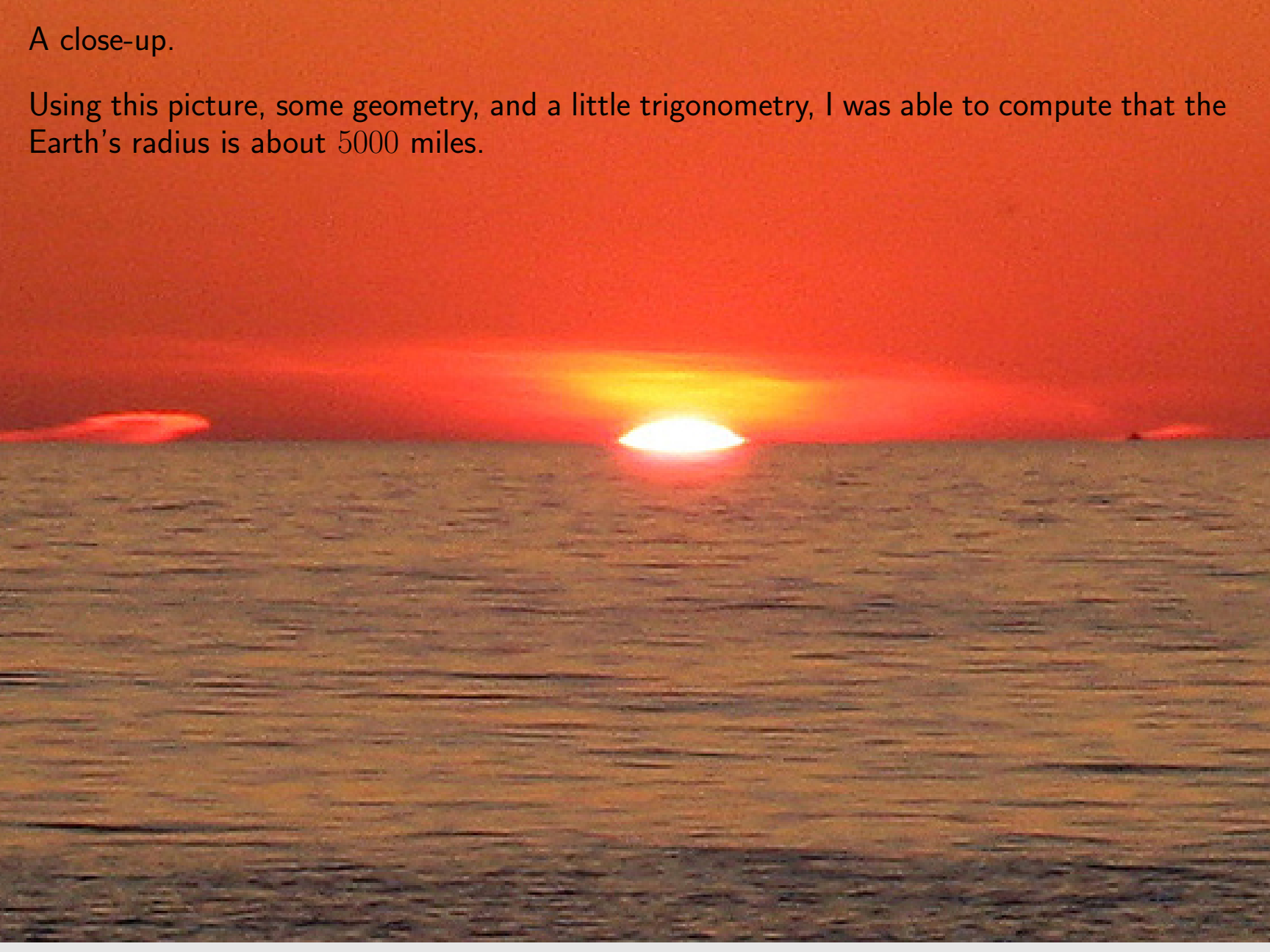
IS THERE AN EASIER WAY?

A picture I took of a sunset over Lake Michigan.



A close-up.

Using this picture, some geometry, and a little trigonometry, I was able to compute that the Earth's radius is about 5000 miles.

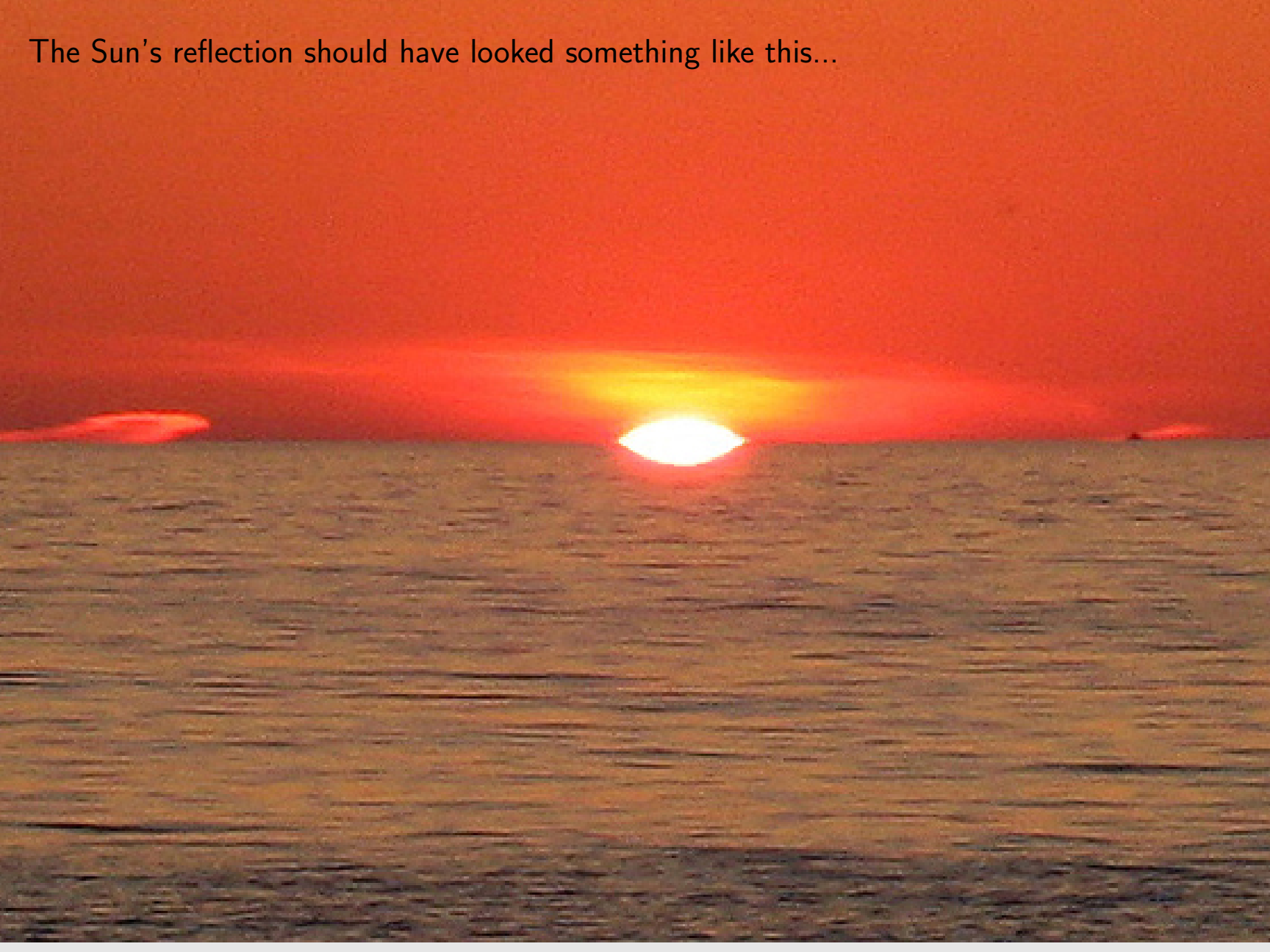


A smooth lake is supposed to act like a mirror.



Credit: Lorene Lavora

The Sun's reflection should have looked something like this...



Or not!

What's going on?

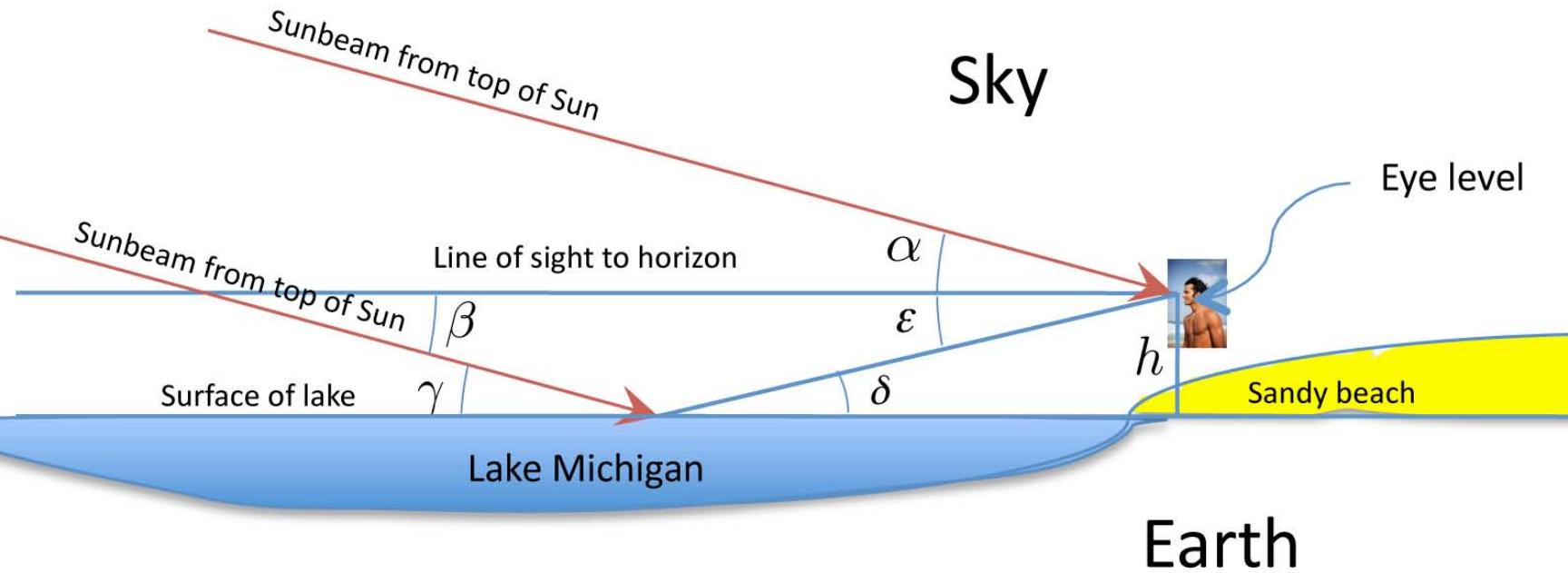
Lake Michigan is not a flat mirror.

Its surface is curved because the Earth is a sphere.

That's why we can't see the shore on the opposite side—it's below the horizon!



# Geometry — If the Earth Were Flat!



- $\alpha = \beta$  alternate interior angles are equal
- $\beta = \gamma$  alternate interior angles are equal
- $\gamma = \delta$  angle of incidence equals angle of reflection (from Physics!)
- $\delta = \epsilon$  alternate interior angles are equal

Therefore,

$$\alpha = \epsilon.$$

The reflection dips just as far below the horizon as the Sun stands above the horizon.

What are all those funny looking letters?

$A \alpha$	$B \beta$	$\Gamma \gamma$	$\Delta \delta$	$E \epsilon$	$Z \zeta$
Alpha	Beta	Gamma	Delta	Epsilon	Zeta

$H \eta$	$\Theta \theta$	$I \iota$	$K \kappa$	$\Lambda \lambda$	$M \mu$
Eta	Theta	Iota	Kappa	Lambda	Mu

$N \nu$	$\Xi \xi$	$O \omicron$	$\Pi \pi$	$P \rho$	$\Sigma \sigma$
Nu	Xi	Omicron	Pi	Rho	Sigma

$T \tau$	$Y \upsilon$	$\Phi \varphi$	$X \chi$	$\Psi \psi$	$\Omega \omega$
Tau	Upsilon	Phi	Chi	Psi	Omega

# Geometry — The Earth Is Not Flat

Draw a picture.

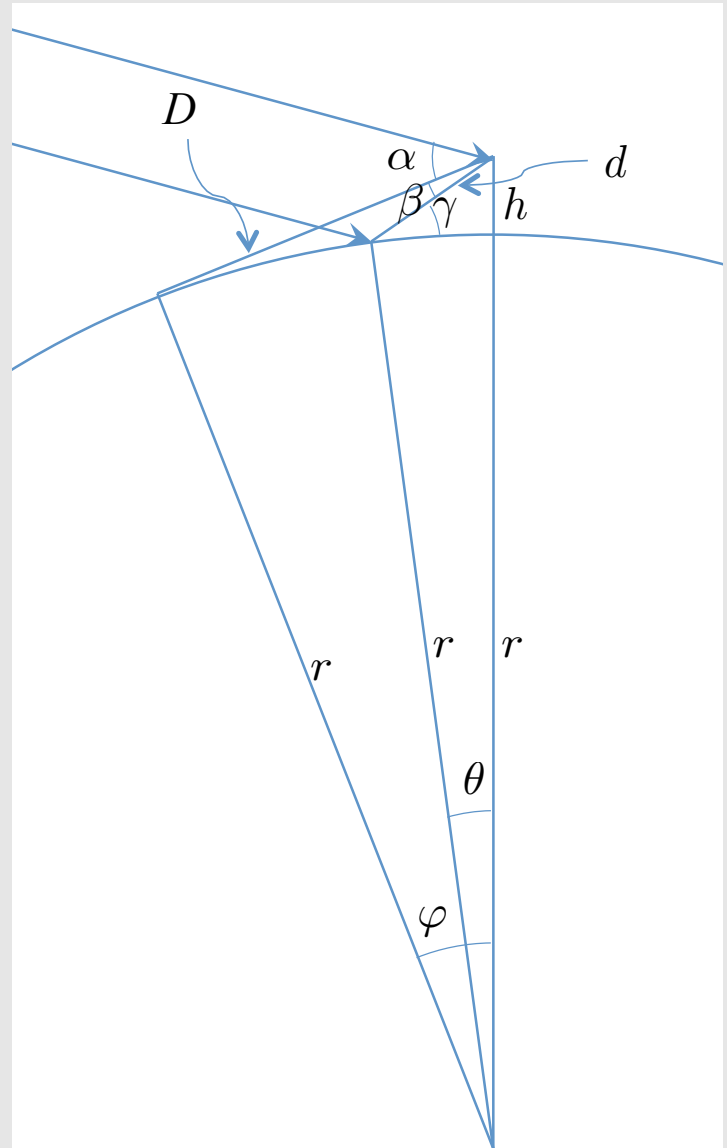
Label everything of possible relevance.

Identify what we know:

$\alpha$  Angle between horizon and top of Sun (measured from photo)

$\beta$  Angle between horizon and “top” of Sun in reflection (measured)

$h$  Height of “eye-level” above “water-level”.



# Geometry — The Earth Is Not Flat

Everything else is the stuff we need to figure out:

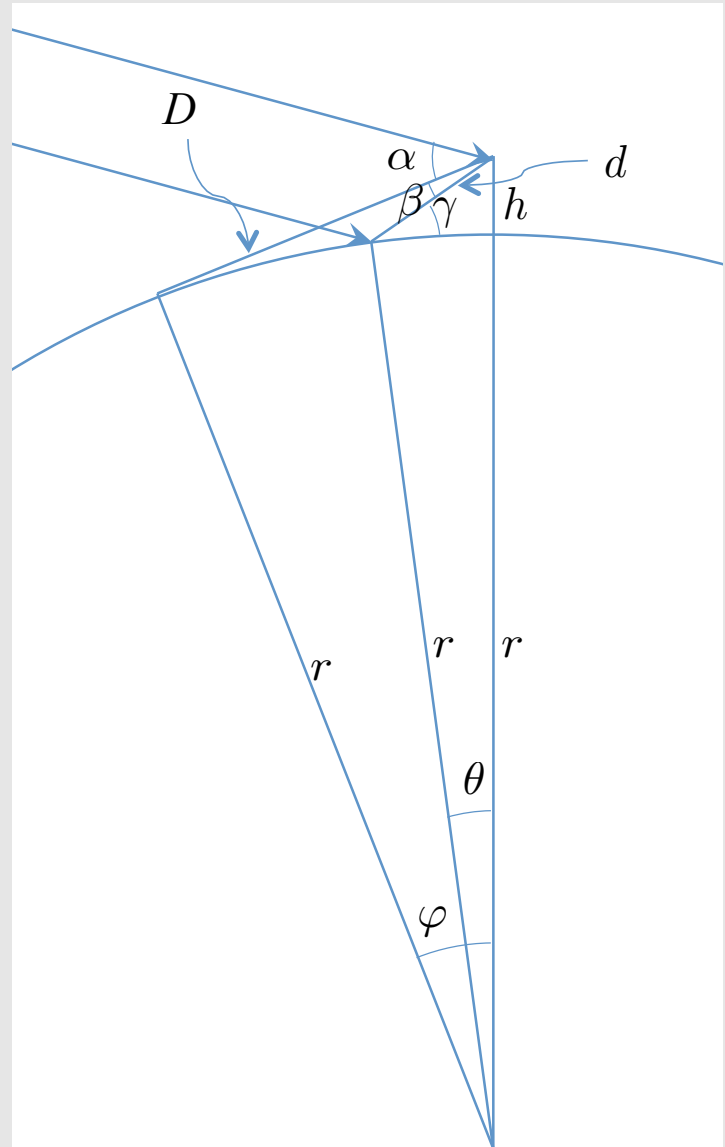
- Three Angles:

- $\gamma$  Angle of reflection off water.

- $\theta$  Angle between observer (me) and point of reflection.

- $\varphi$  Angle between observer (me) and point of horizon.

Plus...



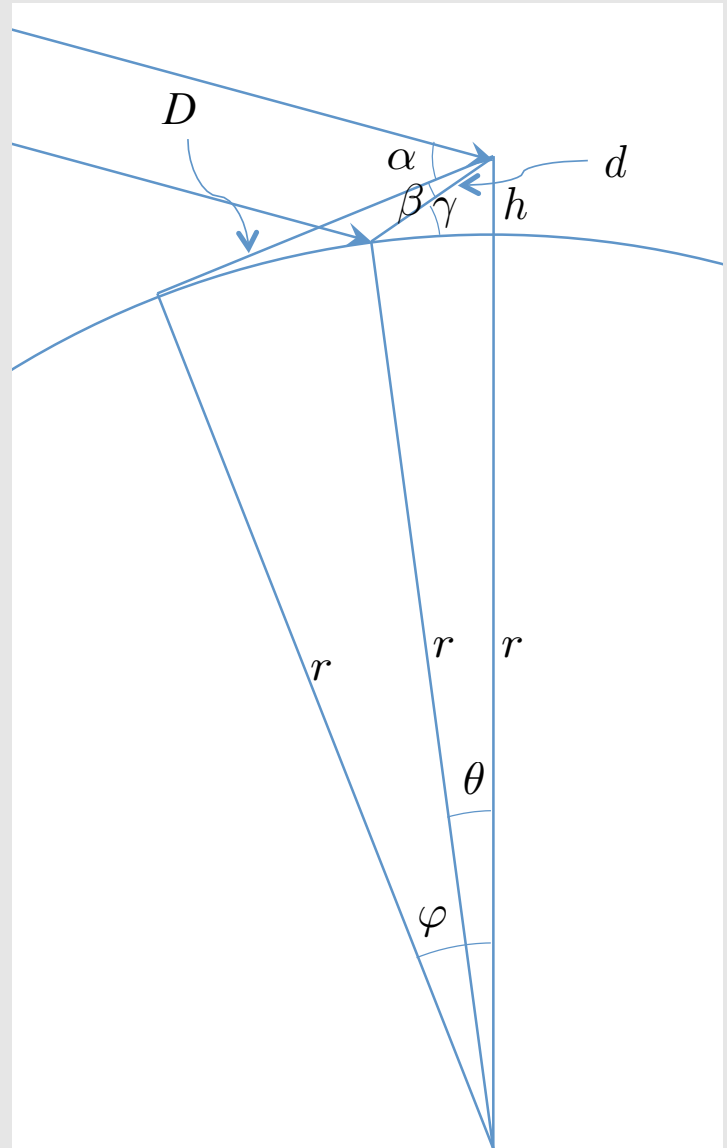
# Geometry — The Earth Is Not Flat

- Three Distances (lengths):

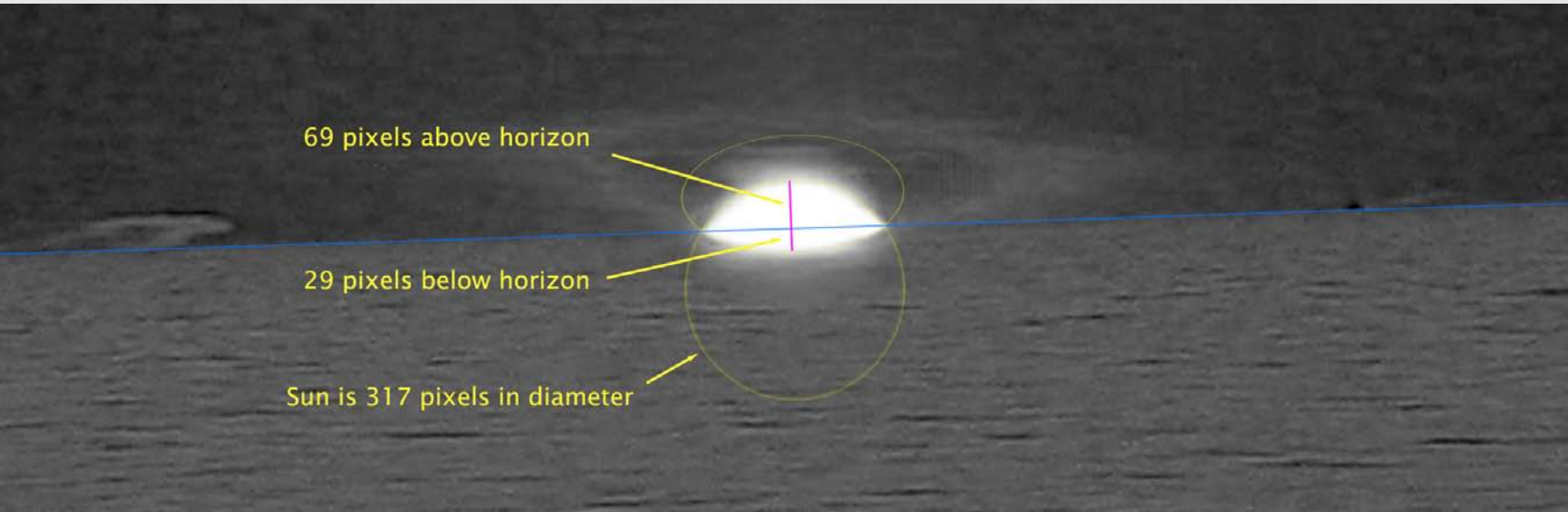
$d$  Distance to point of reflection.

$D$  Distance to horizon.

$r$  Radius of Earth.  $\Leftarrow$  This one is key!!!



## What We Know (Measure!)



The Sun is  $1/2^\circ$  in diameter. Therefore,  $1^\circ$  equals  $2 \times 317 = 634$  pixels.

And so,

$$\alpha = 69 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.1088 \text{ degrees}$$

and

$$\beta = 29 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.0457 \text{ degrees.}$$

And, we assume that eye level is

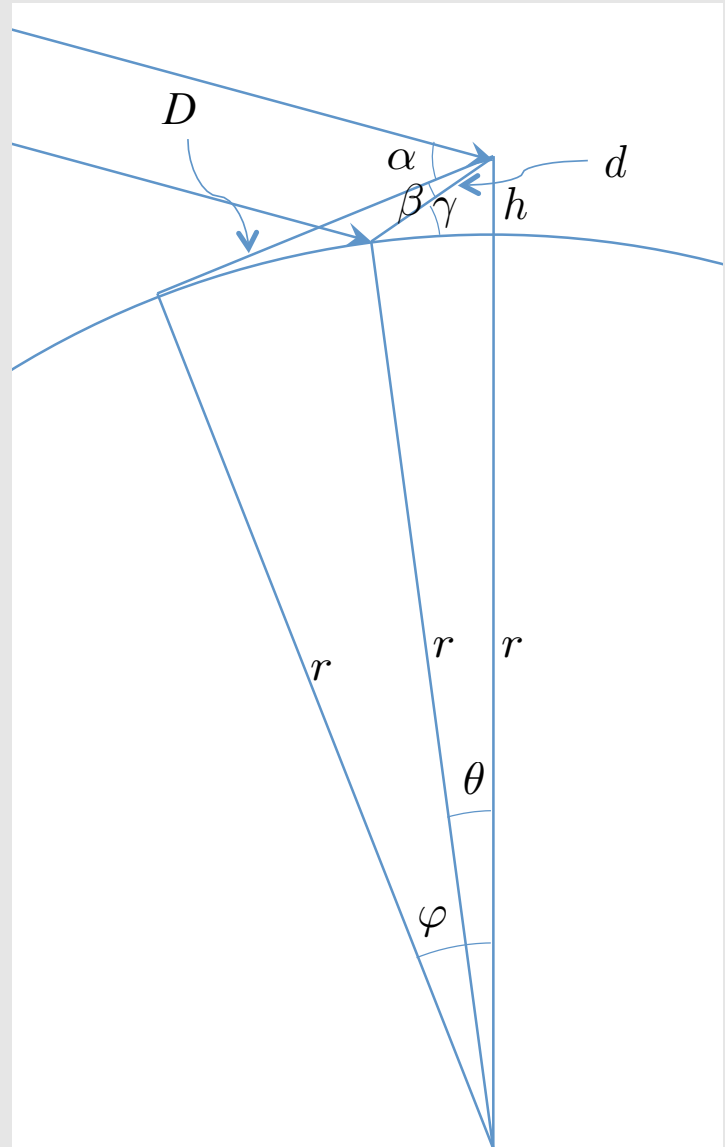
$$h = 7 \text{ feet}$$

## What We Need To Figure Out:

- Angles:
  - $\gamma$  Angle of reflection off water.
  - $\theta$  Angle between observer (me) and point of reflection.
  - $\varphi$  Angle between observer (me) and point of horizon.
- Distances (lengths):
  - $d$  Distance to point of reflection.
  - $D$  Distance to horizon.
  - $r$  Radius of Earth.  $\Leftarrow$  This one is key!!!

That's SIX UNKNOWNNS.

We need SIX (distinct!) EQUATIONS.



Equation 1:

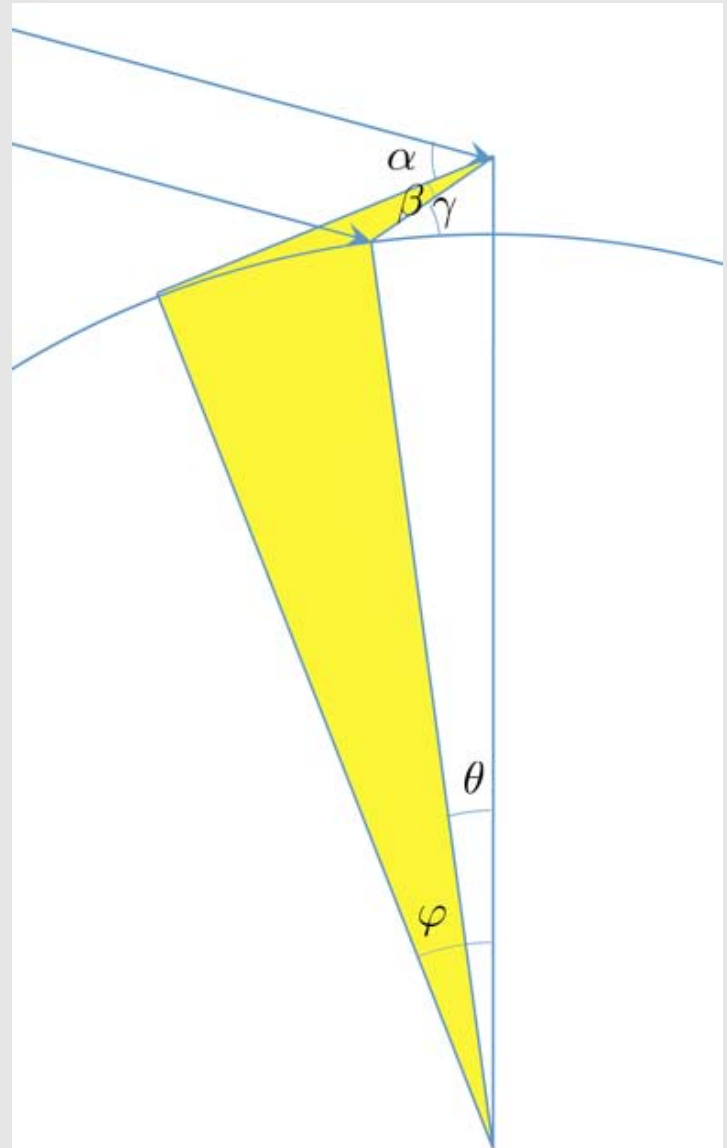
The sum of the angles around a quadrilateral is  $360^\circ$ .

Hence,

$$(\varphi - \theta) + 90 + \beta + (270 - \gamma) = 360.$$

Simplifying, we get

$$\varphi + \beta = \theta + \gamma.$$



Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to  $180^\circ$ .

Hence,

$$\alpha + \beta + \sigma = 180.$$

Also, because angle of incidence equals angle of reflection, we see that

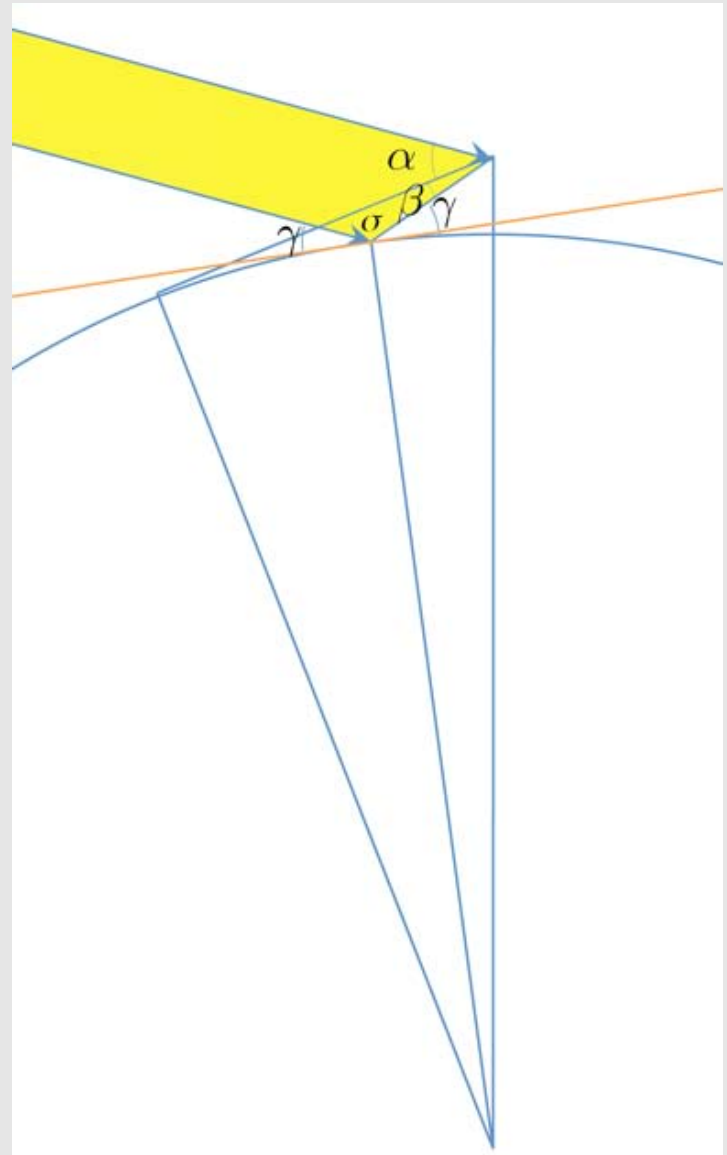
$$\gamma + \sigma + \gamma = 180.$$

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$

So, our second equation is

$$\alpha + \beta = 2\gamma.$$



Equation 3:

The distance from the center of the Earth to eye level is

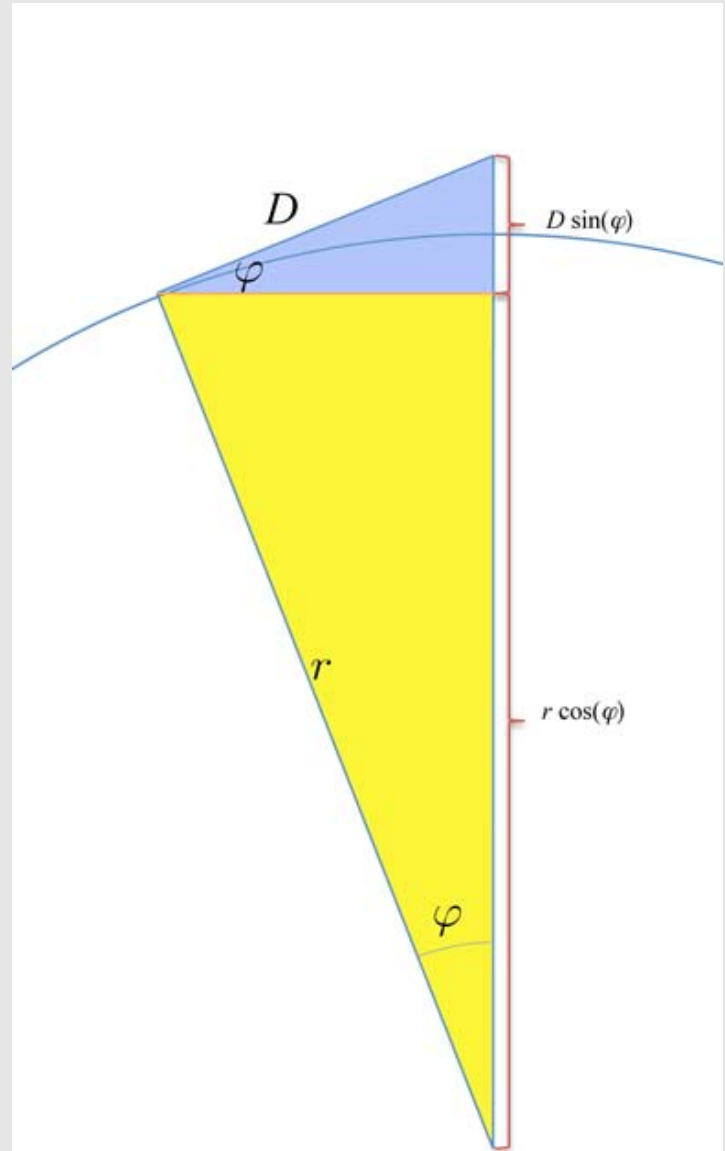
$$r + h.$$

But, it is also

$$D \sin(\varphi) + r \cos(\varphi).$$

Hence,

$$D \sin(\varphi) + r \cos(\varphi) = r + h.$$



Equation 4:

The “horizontal” distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

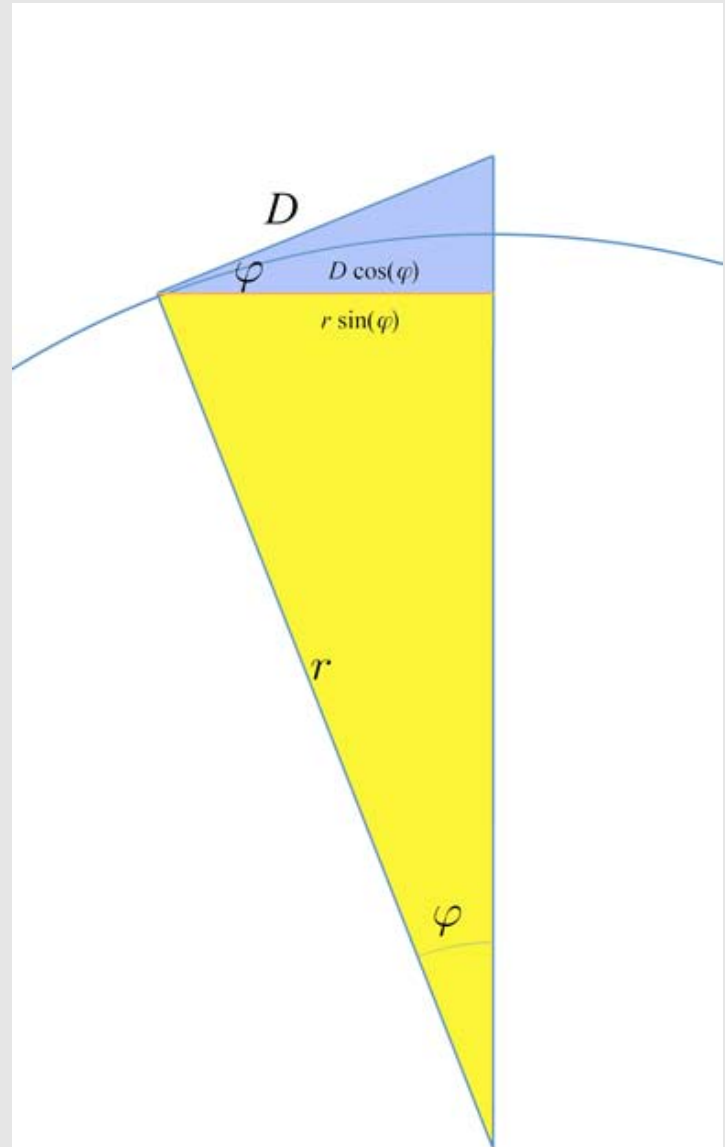
$$D \cos(\varphi)$$

and

$$r \sin(\varphi).$$

Hence,

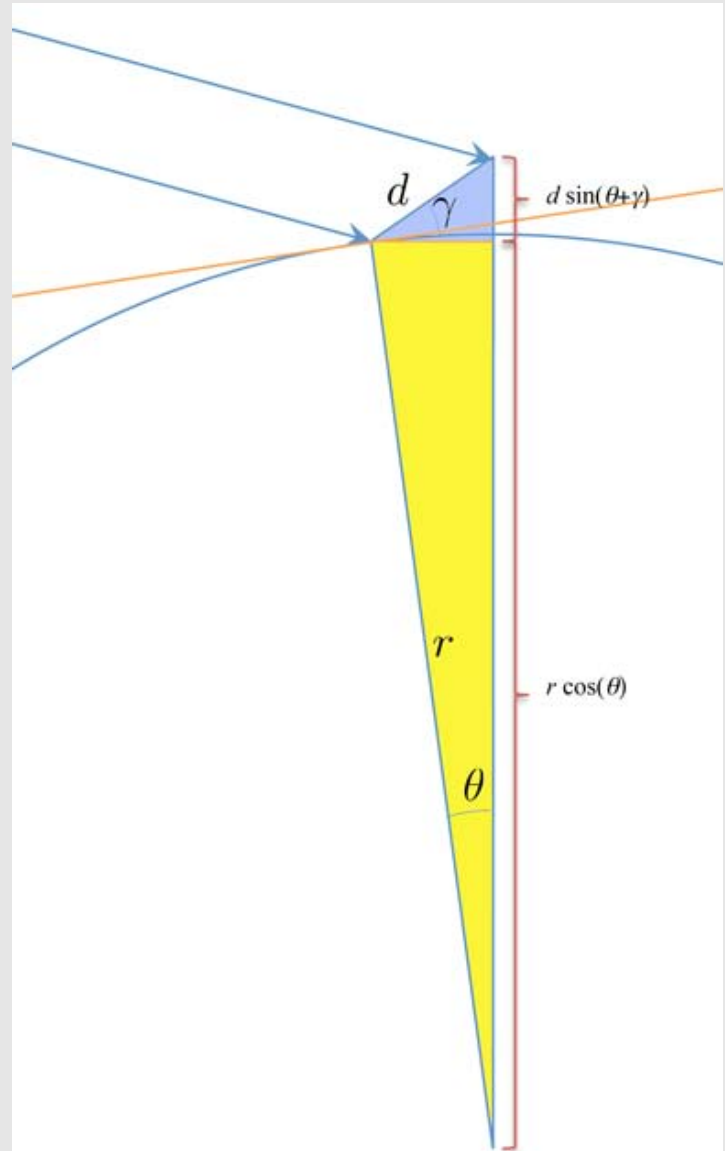
$$D \cos(\varphi) = r \sin(\varphi).$$



Equation 5:

Equation 5 is analogous to Equation 3, using the “point of reflection” in place of the “horizon”.

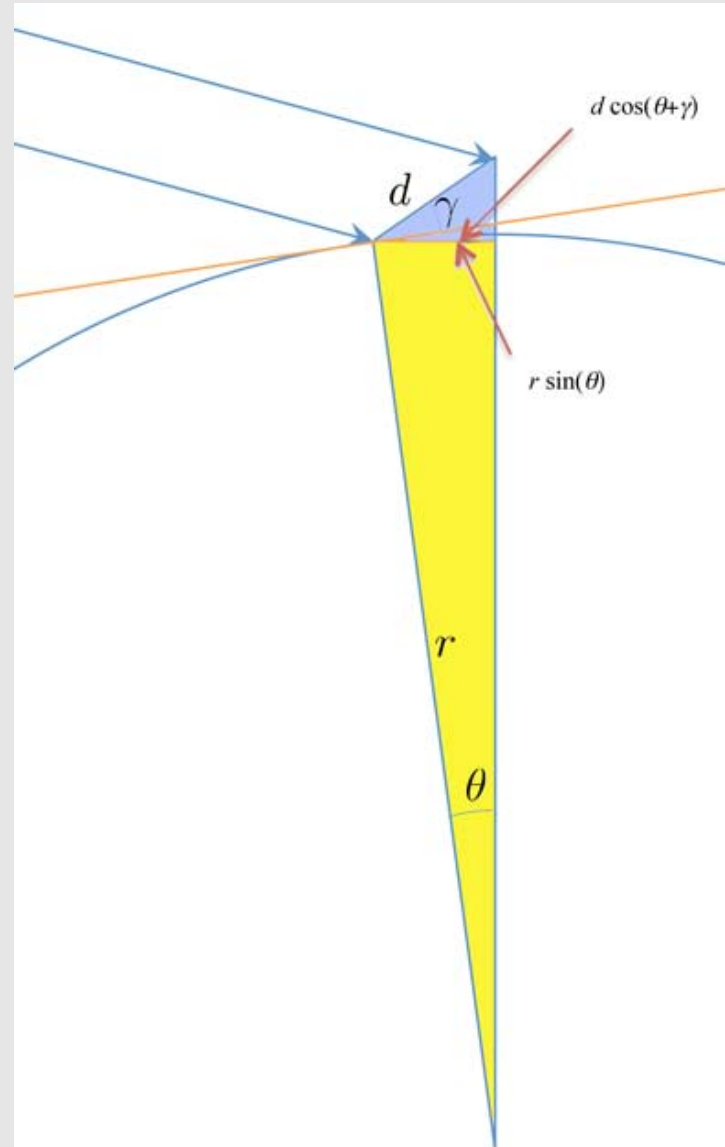
$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h.$$



Equation 6:

Equation 6 is analogous to Equation 4 in the same way.

$$d \cos(\theta + \gamma) = r \sin(\theta).$$



## Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \quad (1)$$

$$\alpha + \beta = 2\gamma \quad (2)$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \quad (3)$$

$$D \cos(\varphi) = r \sin(\varphi) \quad (4)$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \quad (5)$$

$$d \cos(\theta + \gamma) = r \sin(\theta). \quad (6)$$

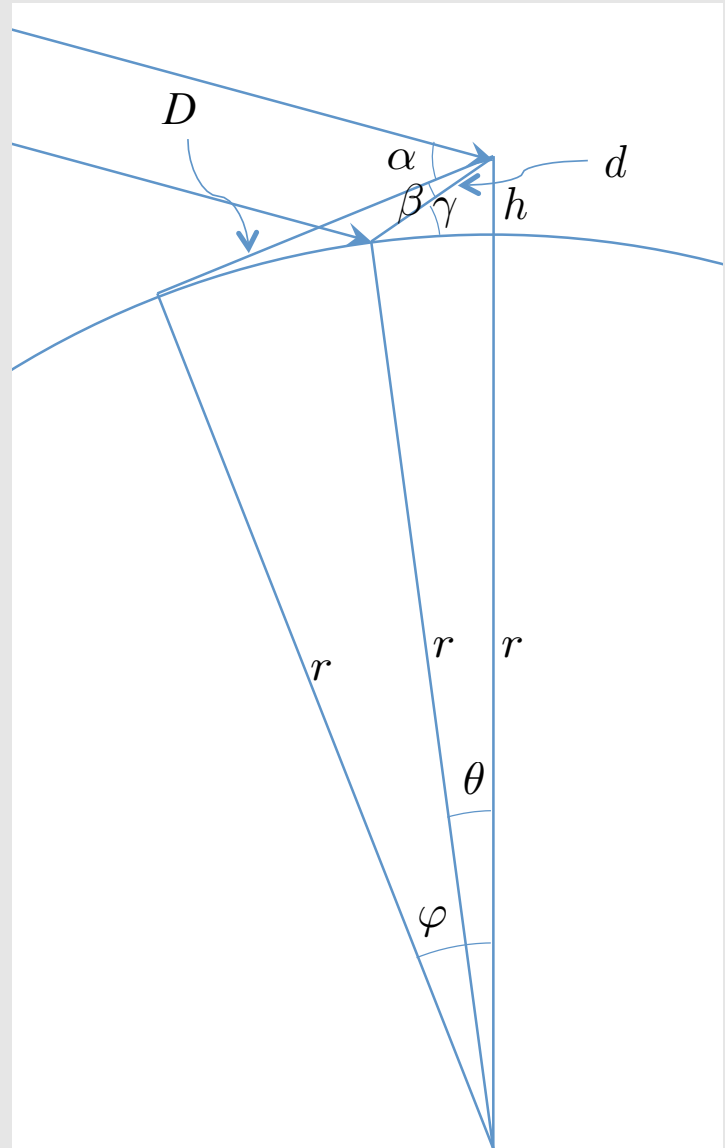
Not hard to solve.

Use (2) to solve for  $\gamma$ .

Solve (4) for  $D$  and then substitute in for  $D$  in (3).

Solve (6) for  $d$  and then substitute in for  $d$  in (5).

And so on...



Answer for  $r$  (radius of Earth) is:

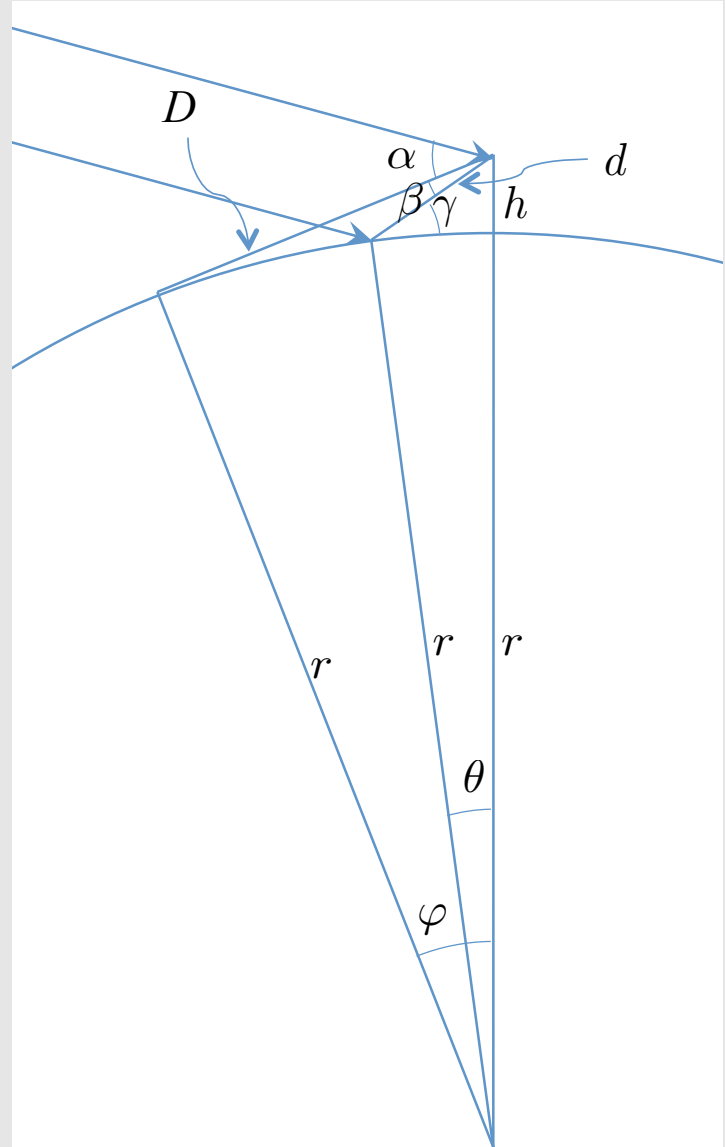
$$r = \frac{h}{\frac{\sqrt{1 - 2 \cos \beta \cos \gamma + \cos^2 \gamma}}{\sin \beta} - 1}$$

where

$$\gamma = \frac{\alpha + \beta}{2}.$$

Plugging in our values for  $\alpha$ ,  $\beta$ , and  $h$ , we get

$$r = 4977 \text{ miles.}$$



### *Morals:*

- Always be mindful of units.
- Always draw a picture and label things.
- If there are six unknowns, you need six (distinct) equations.
- A picture need not be to scale; it can exaggerate angles, distances, etc.
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*Conclusion:* GEOMETRY IS BOTH FUN AND USEFUL.