Making Dark Shadows with Linear Programming

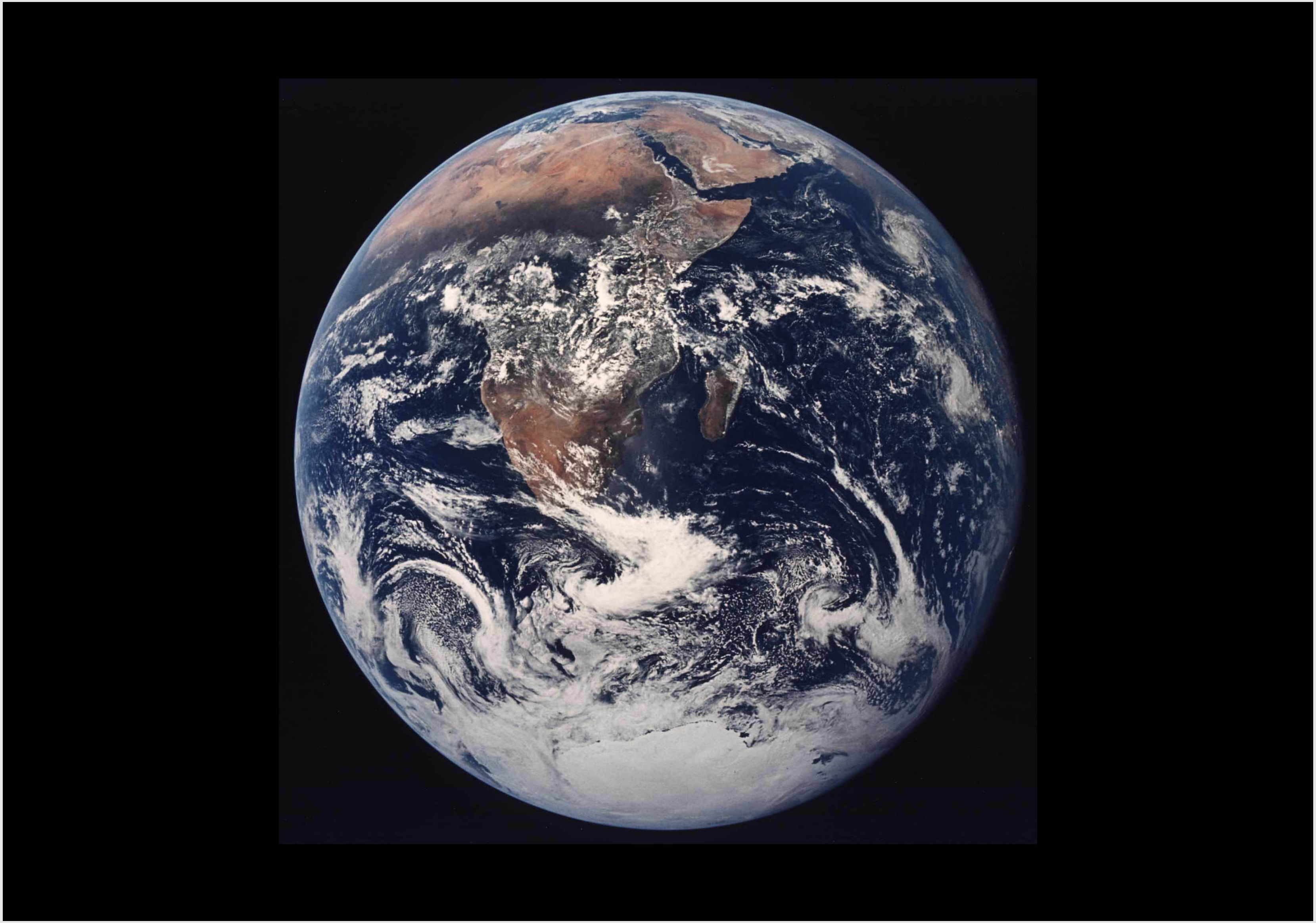
Robert J. Vanderbei

2008 Nov 10

Faculty of Engineering
Dept. of Management Sciences
University of Waterloo

http://www.princeton.edu/~rvdb

Are We Alone?



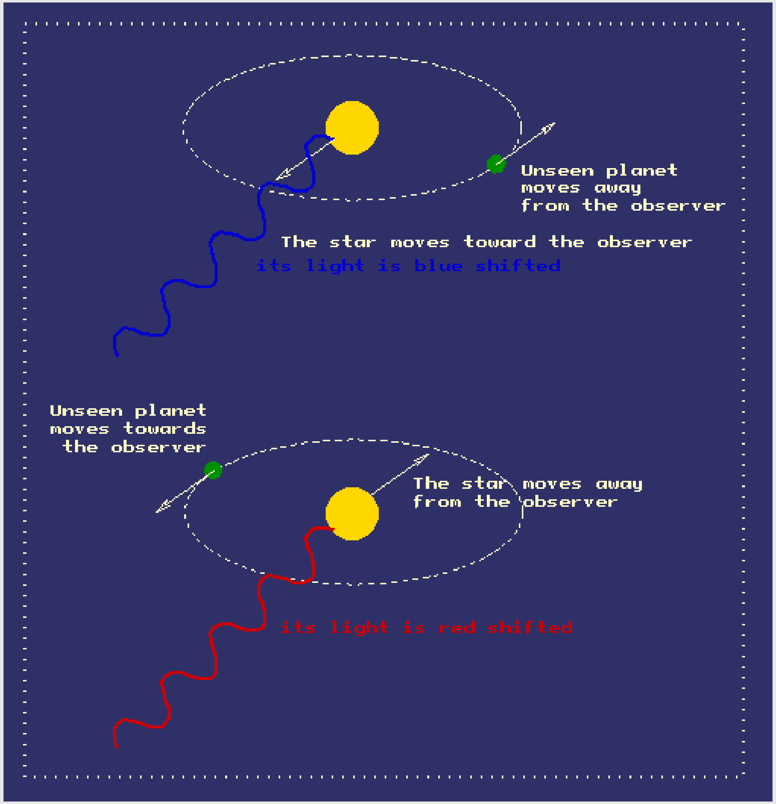
Indirect Detection Methods

Over 300 planets found—more all the time

Wobble Methods

Radial Velocity.
For edge-on systems.
Measure periodic doppler shift.

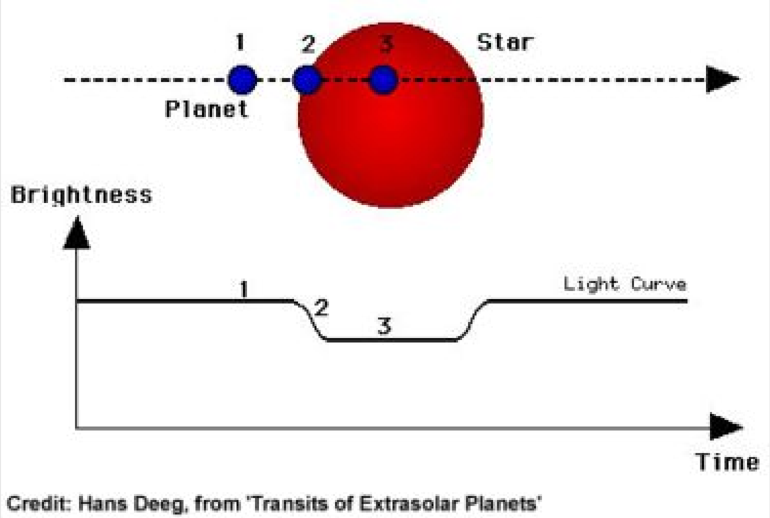
Astrometry.
Best for face-on systems.
Measure circular wobble against background stars.



Transit Method

- HD209458b confirmed both via RV and transit.
- Period: 3.5 days
- Separation: 0.045 AU (0.001 arcsecs)
- Radius: $1.3R_J$
- Intensity Dip: $\sim 1.7\%$
- Venus Dip $= 0.01\%$, Jupiter Dip: 1%

Venus Transit (R.J. Vanderbei)



Terrestrial Planet Finder Telescope (TPF)

- DETECT: Search 150-500 nearby (5-15 pc distant) Sun-like stars for Earth-like planets.
- CHARACTERIZE: Determine basic physical properties and measure "biomarkers", indicators of life or conditions suitable to support it.

Why Is It Hard? Can't Hubble do it?

- If the star is Sun-like and the planet is Earth-like, then the reflected visible light from the planet is 10^{-10} times as bright as the star. This is a difference of 25 magnitudes!
- If the star is 10 pc (33 ly) away and the planet is 1 AU from the star, the angular separation is 0.1 arcseconds!
- A point source (i.e. star) produces not a point image but an *Airy pattern* consisting of an *Airy disk* surrounded by a system of *diffraction rings* completely covering the nearby planet.
- By *apodizing* the entrance pupil, one can control the shape and strength of the diffraction rings.

Electric Field

The image-plane *electric field* $E()$ produced by an on-axis plane wave and an apodized aperture defined by an *apodization function* $A()$ is given by

$$
\begin{aligned}
E(\xi, \zeta) &= \iint_{\bigcirc} e^{i(x\xi + y\zeta)} A(x, y) dy dx \\
&\vdots \\
E(\rho) &= 2\pi \int_0^{1/2} J_0(r\rho) A(r) r dr,
\end{aligned}
$$

where J_0 denotes the 0-th order Bessel function of the first kind.

NOTE: The *electric field* depends *linearly* on the *apodization function*.

The unitless pupil-plane "length" r is given as a multiple of the aperture D.

The unitless image-plane "length" ρ is given as a multiple of focal-length times wavelength over aperture $(f\lambda/D)$ or, equivalently, as an angular measure on the sky, in which case it is a multiple of just λ/D. (Example: $\lambda = 0.5\mu$m and $D = 10$m implies $\lambda/D = 10$mas.)

The *intensity* is the square of the electric field.

Performance Metrics

Inner and Outer Working Angles

$$\rho_{\mathsf{iwa}} \qquad \rho_{\mathsf{owa}}$$

Contrast:

$$E^2(\rho)/E^2(0)$$

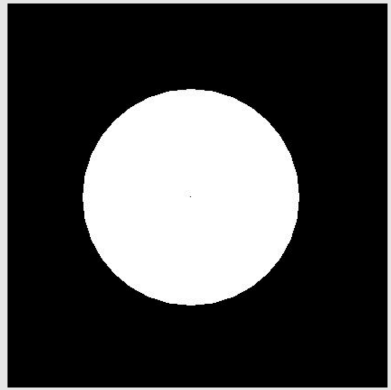
Throughput:

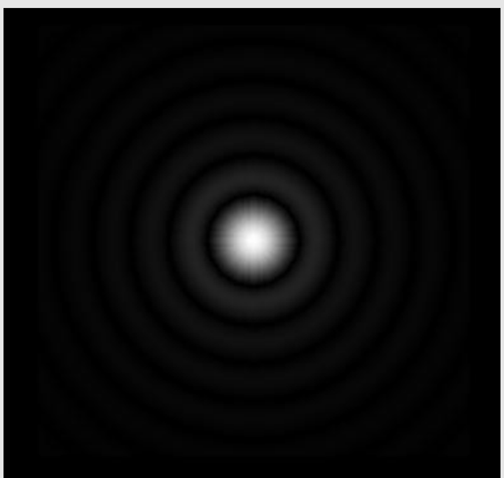
$$\mathcal{T}_{\mathsf{Airy}} = \frac{\displaystyle\int_0^{\rho_{\mathsf{iwa}}} E^2(\rho) 2\pi\rho d\rho}{(\pi(1/2)^2)} = 8\int_0^{\rho_{\mathsf{iwa}}} E^2(\rho)\rho d\rho.$$

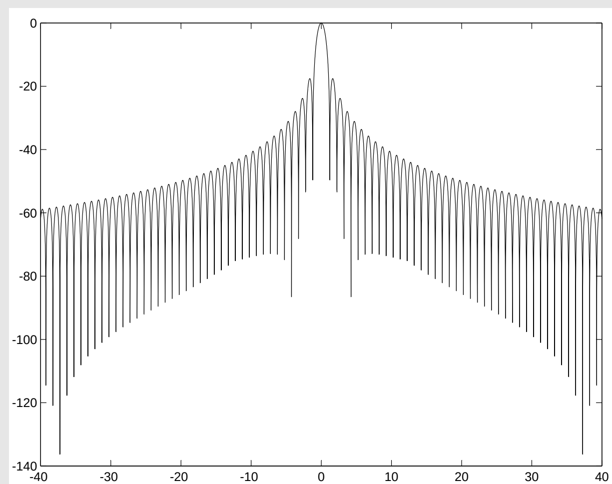
Clear Aperture—Airy Pattern

$\rho_{\text{iwa}} = 1.24 \quad \mathcal{T}_{\text{Airy}} = 84.2\% \quad \text{Contrast} = 10^{-2}$

$\rho_{\text{iwa}} = 748 \quad \mathcal{T}_{\text{Airy}} = 100\% \quad \text{Contrast} = 10^{-10}$



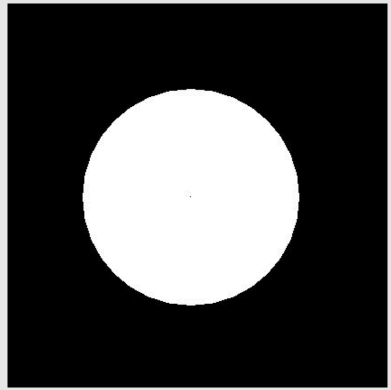


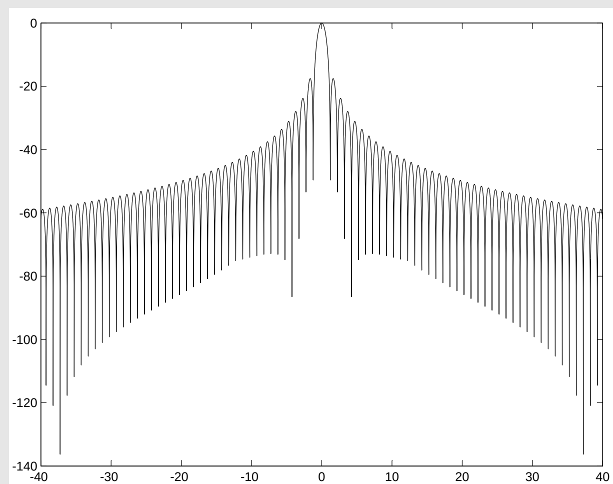


Clear Aperture—Airy Pattern

$$\rho_{\mathrm{iwa}} = 1.24 \quad \mathcal{T}_{\mathrm{Airy}} = 84.2\% \quad \mathrm{Contrast} = 10^{-2}$$

$$\rho_{\mathrm{iwa}} = 748 \quad \mathcal{T}_{\mathrm{Airy}} = 100\% \quad \mathrm{Contrast} = 10^{-10}$$





Optimization

Find *apodization* function $A()$ that solves:

$$\begin{array}{lll} \text{maximize} & \displaystyle\int_0^{1/2} A(r) 2\pi r dr & \\ \text{subject to} & -10^{-5} E(0) \leq E(\rho) \leq 10^{-5} E(0), & \rho_{\text{iwa}} \leq \rho \leq \rho_{\text{owa}}, \\ & 0 \leq A(r) \leq 1, & 0 \leq r \leq 1/2, \end{array}$$

Optimization

Find *apodization* function $A()$ that solves:

$$
\begin{aligned}
\text{maximize} \quad & \int_0^{1/2} A(r) 2\pi r dr \\
\text{subject to} \quad & -10^{-5} E(0) \le E(\rho) \le 10^{-5} E(0), && \rho_{\text{iwa}} \le \rho \le \rho_{\text{owa}}, \\
& 0 \le A(r) \le 1, && 0 \le r \le 1/2, \\
& -50 \le A''(r) \le 50, && 0 \le r \le 1/2
\end{aligned}
$$

An infinite dimensional *linear programming* problem.

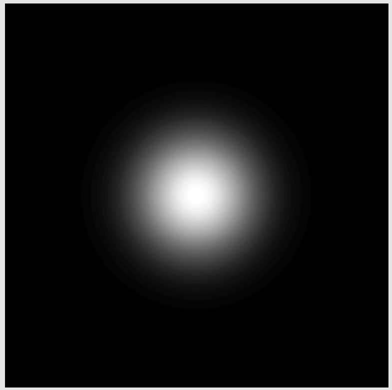
The AMPL Model

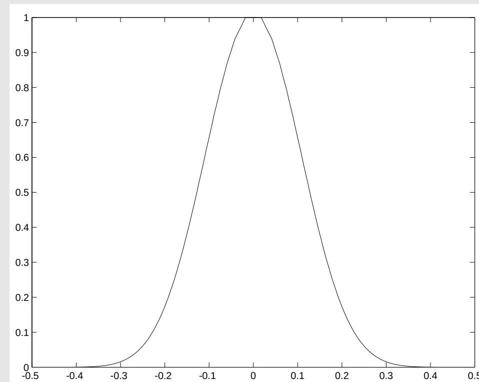
```
function J0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

param dr := (1/2)/N;
set Rs ordered := setof {j in 0.5..N-0.5 by 1} (1/2)*j/N;

var A {Rs} >= 0, <= 1, := 1/2;

set Rhos ordered := setof {j in 0..N} j*rho1/N;
set PlanetBand := setof {rho in Rhos: rho>=rho0 && rho<=rho1} rho;

var E0 {rho in Rhos} =
    2*pi*sum {r in Rs} A[r]*J0(2*pi*r*rho)*r*dr;

maximize area: sum {r in Rs} 2*pi*A[r]*r*dr;
subject to sidelobe_pos {rho in PlanetBand}: E0[rho] <= 10^(-5)*E0[0];
subject to sidelobe_neg {rho in PlanetBand}: -10^(-5)*E0[0] <= E0[rho];

subject to smooth {r in Rs: r != first(Rs) && r != last(Rs)}:
    -50*dr^2 <= A[next(r)] - 2*A[r] + A[prev(r)] <= 50*dr^2;

solve;
```

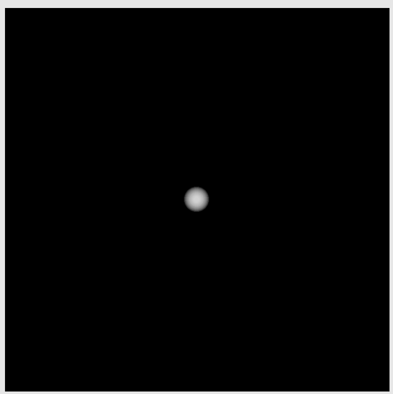
The Optimal Apodization

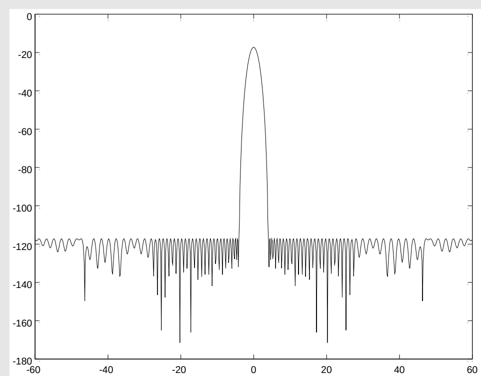
$$\rho_{\text{iwa}} = 4 \quad \mathcal{T}_{\text{Airy}} = 9\%$$

Excellent dark zone. Unmanufacturable.









Concentric Ring Masks

Recall that for circularly symmetric apodizations

$$E(\rho) = 2\pi \int_0^{1/2} J_0(r\rho)A(r)rdr,$$

where J_0 denotes the 0-th order Bessel function of the first kind.

Let

$$A(r) = \begin{cases} 1 & r_{2j} \le r \le r_{2j+1}, \qquad j = 0, 1, \ldots, m-1 \\ 0 & \text{otherwise}, \end{cases}$$

where

$$0 \le r_0 \le r_1 \le \cdots \le r_{2m-1} \le 1/2.$$

The integral can now be written as a sum of integrals and each of these integrals can be explicitly integrated to get:

$$E(\rho) \;=\; \sum_{j=0}^{m-1} \frac{1}{\rho} \Big(r_{2j+1} J_1\left(\rho r_{2j+1}\right) - r_{2j} J_1\left(\rho r_{2j}\right) \Big).$$

Mask Optimization Problem

$$\text{maximize} \sum_{j=0}^{m-1} \pi(r_{2j+1}^2 - r_{2j}^2)$$

$$\text{subject to:} \quad -10^{-5}E(0) \le E(\rho) \le 10^{-5}E(0), \quad \text{for } \rho_0 \le \rho \le \rho_1$$

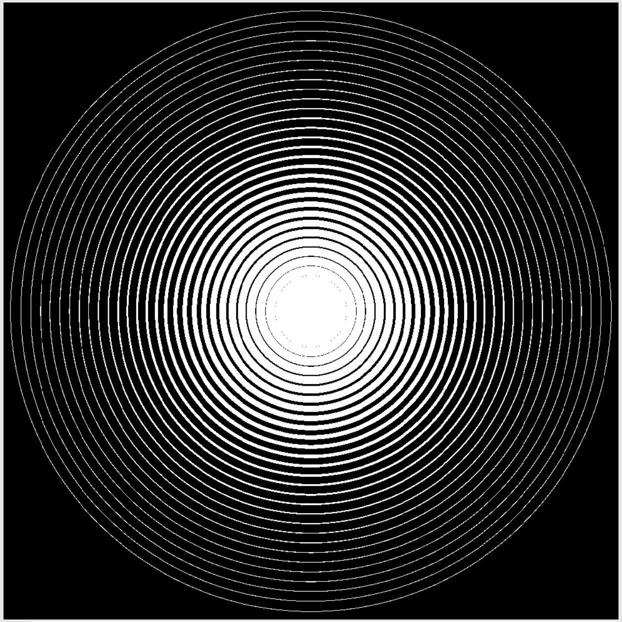
where $E(\rho)$ is the function of the r_j's given on the previous slide.

This problem is a semiinfinite nonconvex optimization problem.

The AMPL Model

```
function intrJ0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

var r {j in 0..M} >= 0, <= 1/2, := r0[j];

set Rhos2 ordered := setof {j in 0..N} (j+0.5)*rho1/N;
set PlanetBand2 := setof {rho in Rhos2: rho>=rho0 && rho<=rho1} rho;

var E {rho in Rhos2} =
    (1/(2*pi*rho)^2)*sum {j in 0..M by 2}
(intrJ0(2*pi*rho*r[j+1]) - intrJ0(2*pi*rho*r[j]));

maximize area2: sum {j in 0..M by 2} (pi*r[j+1]^2 - pi*r[j]^2);
subject to sidelobe_pos2 {rho in PlanetBand2}: E[rho] <= 10^(-5)*E[first(rhos2)];
subject to sidelobe_neg2 {rho in PlanetBand2}: -10^(-5)*E[first(rhos2)] <= E[rho];

subject to order {j in 0..M-1}: r[j+1] >= r[j];

solve mask;
```

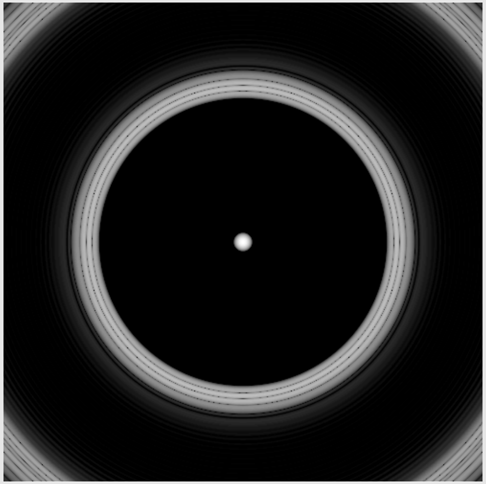
The Best Concentric Ring Mask

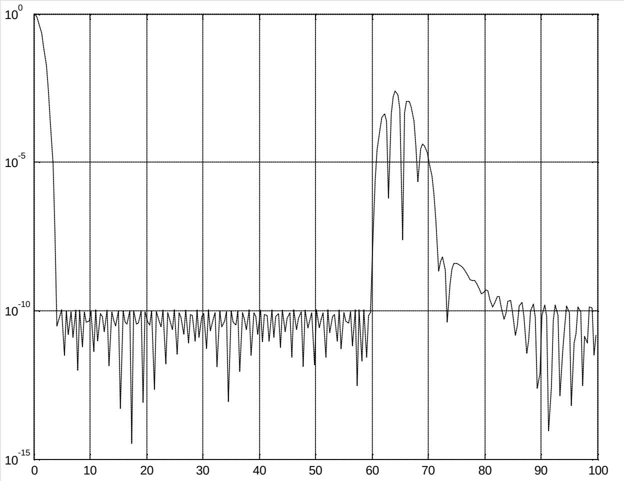


$$\rho_{\mathrm{iwa}} = 4 \quad \rho_{\mathrm{owa}} = 60$$

$$\mathcal{T}_{\mathrm{Airy}} = 9\%$$

Lay it on glass?





Other Masks

Consider a binary apodization (i.e., a mask) consisting of an opening given by

$$A(x,y) = \begin{cases} 1 & |y| \leq a(x) \\ 0 & \text{else} \end{cases}$$

We only consider masks that are symmetric with respect to both the x and y axes. Hence, the function $a()$ is a nonnegative even function.

In such a situation, the electric field $E(\xi, \zeta)$ is given by

$$\begin{aligned} E(\xi,\zeta) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-a(x)}^{a(x)} e^{i(x\xi + y\zeta)} dy dx \\ &= 4 \int_0^{\frac{1}{2}} \cos(x\xi) \frac{\sin(a(x)\zeta)}{\zeta} dx \end{aligned}$$

Maximizing Throughput

Because of the symmetry, we only need to optimize in the first quadrant:

$$\text{maximize } 4\int_0^{\frac{1}{2}} a(x)dx$$

$$\begin{aligned} \text{subject to } & -10^{-5}E(0,0) \leq E(\xi,\zeta) \leq 10^{-5}E(0,0), && \text{for } (\xi,\zeta) \in \mathcal{O} \\ & 0 \leq a(x) \leq 1/2, && \text{for } 0 \leq x \leq 1/2 \end{aligned}$$

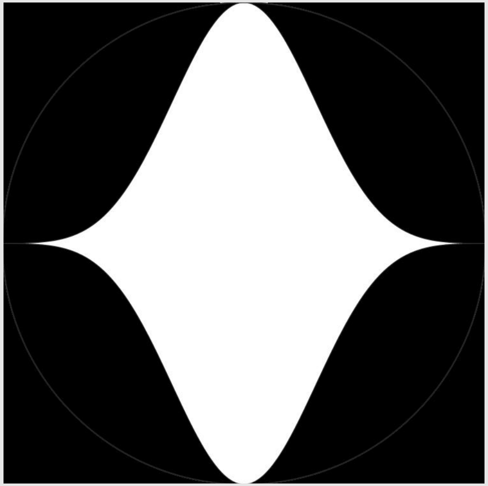
The objective function is the total open area of the mask. The first constraint guarantees 10^{-10} light intensity throughout a desired region of the focal plane, and the remaining constraint ensures that the mask is really a mask.

If the set $\mathcal{O}$ is a subset of the x-axis, then the problem is an infinite dimensional linear programming problem.

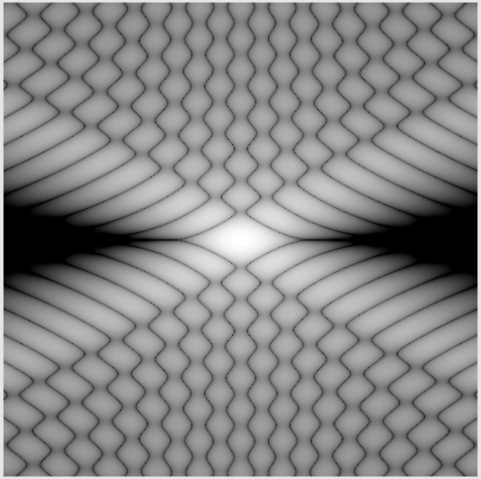
One Pupil w/ On-Axis Constraints

$$\rho_{\text{iwa}} = 4 \quad \mathcal{T}_{\text{Airy}} = 43\%$$

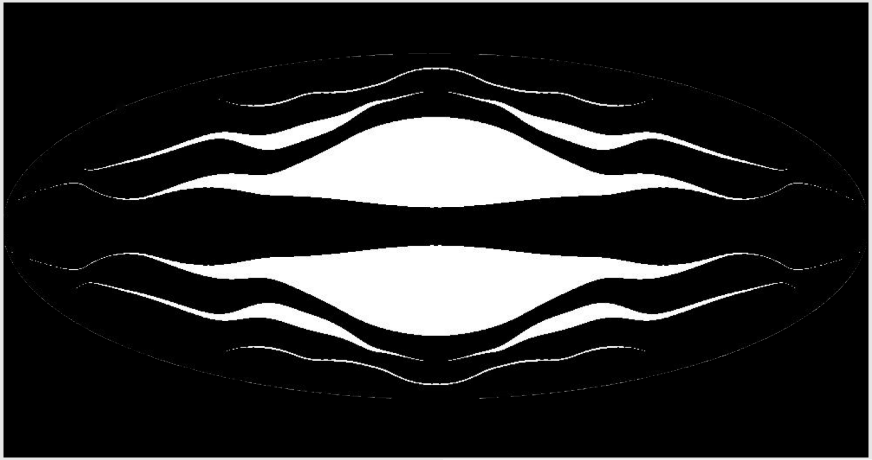
Small dark zone...Many rotations required



PSF for Single Prolate Spheroidal Pupil



Multiple Pupil Mask



$\rho_{\text{iwa}} = 4$

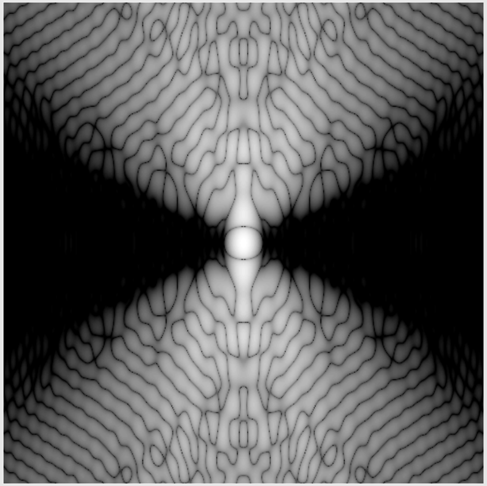
$\mathcal{T}_{\text{Airy}} = 30\%$

Throughput relative to ellipse

11% central obstr.

Easy to make

Only a few rotations



Space Occulter Design

for Planet-Finding

Space-based Occulter (TPF-O)

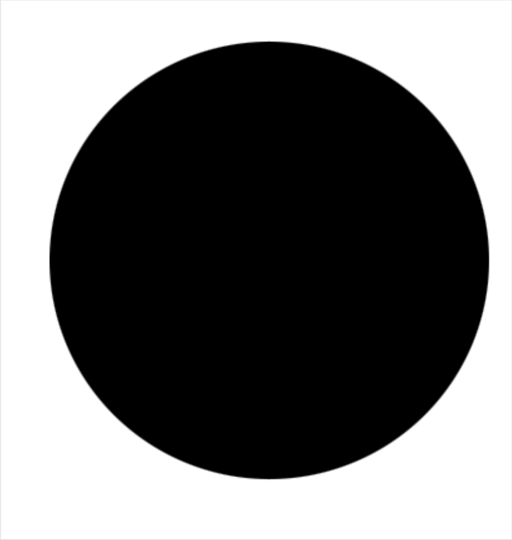
Telescope Aperture: 4m, Occulter Diameter: 50m, Occulter Distance: $72,000$km

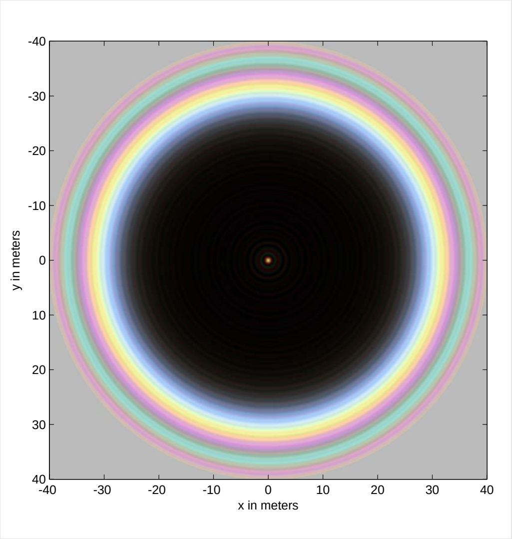
Plain External Occulter (Doesn't Work!)

Shadow ⇒

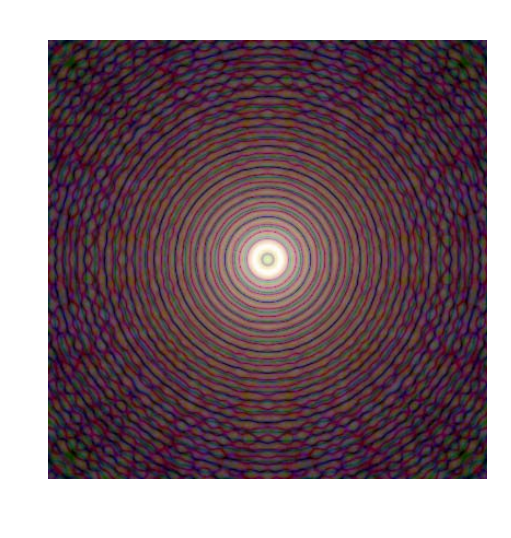
Circular Occulter

⇓



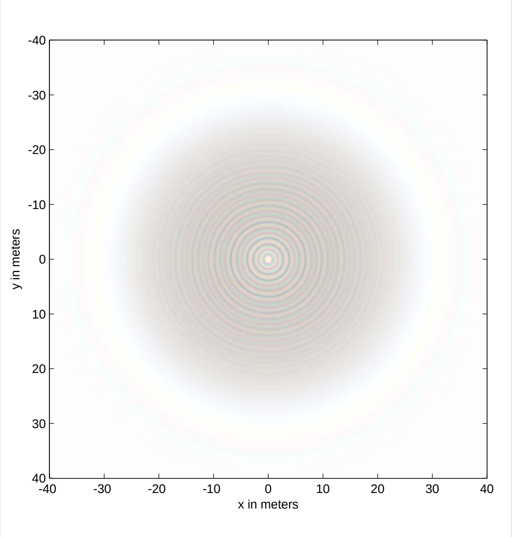


⇐ Note bright spot at center (Poisson's spot)



⇑

Telescope Image



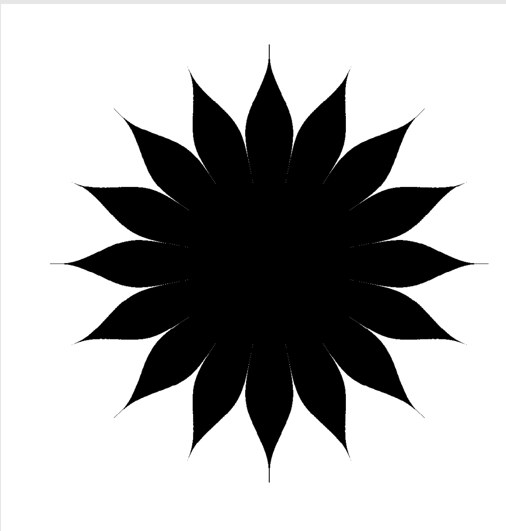
⇐ Shadow (Log Stretch)

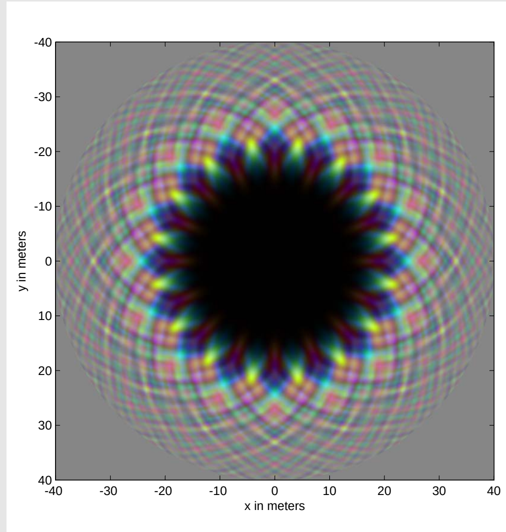
Shaped Occulter—Eliminates Poisson's Spot

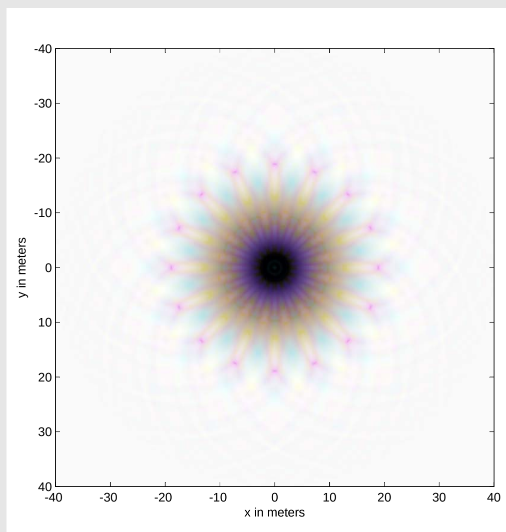
Shadow ⇒

Shaped Occulter

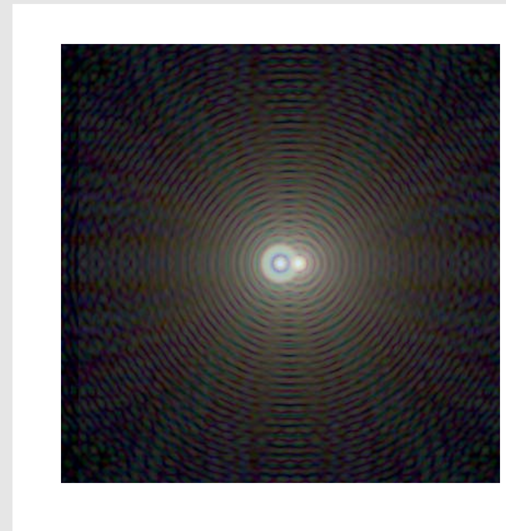
⇓







⇐ Bright spot is gone



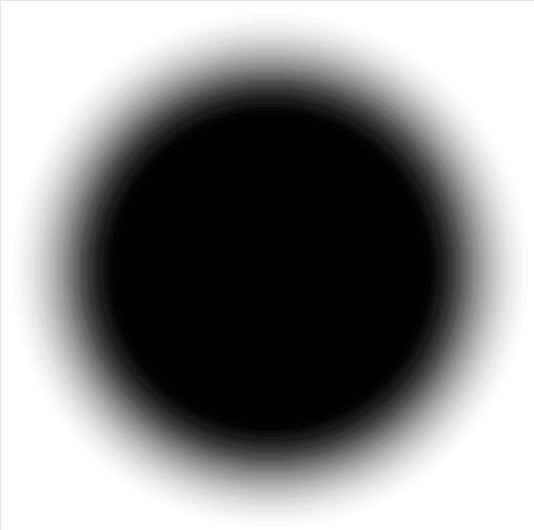
⇑

Telescope image shows planet

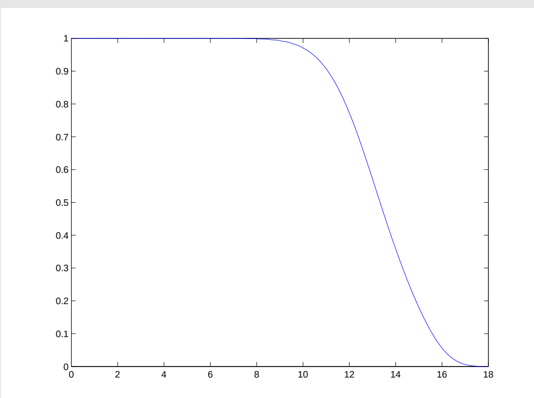
⇐ Shadow is dark

(Log Stretch)

Apodized Occulters



Apodized Occulter



Radial Attenuation $A(r)$

- The problem (as before) is *diffraction*.
- Abrupt edges create unwanted diffraction.
- *Solution:* Soften the edges with a partially transmitting material—an *apodizer*.
- Let $A(r,\theta)$ denote *attenuation* at location (r,θ) on the occulter.
- The *intensity* of the downstream light is given by the *square of the magnitude of the electric field* $E(\rho,\phi)$.
- *Babinet's principle* plus *Fresnel propagation* gives a formula for the downstream electric field:

$$E(\rho,\phi) = 1 - \frac{1}{i\lambda z}\int_0^\infty \int_0^{2\pi} e^{\frac{i\pi}{\lambda z}(r^2+\rho^2-2r\rho\cos(\theta-\phi))} A(r,\theta) r d\theta dr.$$

where

- z is distance "downstream" and
- λ is wavelength of light.

Attenuation Profile Optimization

$$
\begin{array}{llll}
\text{minimize} & & \gamma & \\
\text{subject to} & -\gamma & \leq \Re(E(\rho)) \leq \gamma & \text{for } \rho \in \mathcal{R}, \quad \lambda \in \Lambda \\
& -\gamma & \leq \Im(E(\rho)) \leq \gamma & \text{for } \rho \in \mathcal{R}, \quad \lambda \in \Lambda \\
& & A'(r) \leq 0 & \text{for } 0 \leq r \leq R \\
& -d & \leq A''(r) \leq d & \text{for } 0 \leq r \leq R
\end{array}
$$

Specific choice:

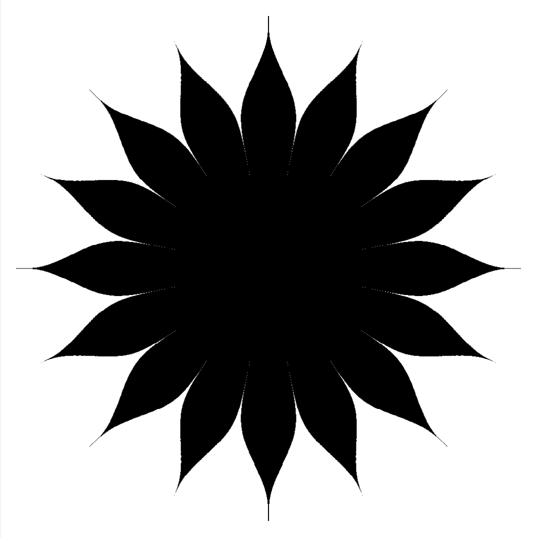
$$R = 25, \quad d = 0.04, \quad \mathcal{R} = [0, 3], \quad \Lambda = [0.4, 1.1] \times 10^{-6}$$

where all metric quantities are in meters.

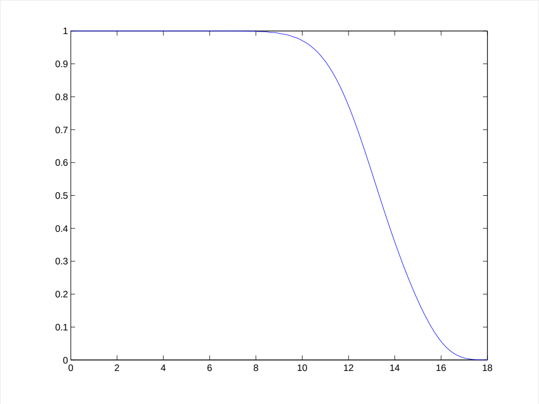
An infinite dimensional linear programming problem.
Discretize:

- $[0, R]$ into 5000 evenly space points.
- $\mathcal{R}$ into 150 evenly spaced points.
- Λ into increments of 0.1×10^{-6}.

Petal-Shaped Occulters



16-Petal Occulter $A(r, \theta)$



Radial Attenuation $A(r)$

- From Jacobi-Anger expansion we get:

$$
\begin{aligned}
E(\rho,\phi) &= 1 - \frac{2\pi}{i\lambda z}\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_0\left(\frac{2\pi r\rho}{\lambda z}\right) A(r) r dr \\
&\quad - \sum_{k=1}^{\infty} \frac{2\pi(-1)^k}{i\lambda z}\left(\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_{kN}\left(\frac{2\pi r\rho}{\lambda z}\right)\frac{\sin(\pi k A(r))}{\pi k} r dr\right) \\
&\quad \times \left(2\cos(kN(\phi - \frac{\pi}{2}))\right)
\end{aligned}
$$

  where N is the number of petals.

- For small ρ, truncated summation well-approximates full sum.
- Truncated after 10 terms.
- $\lambda \in [0.4, 1.1]$ microns.
- $z = 72,000$ km, $R = 25$ m.
- In angular terms, $R/z = 73$ mas.