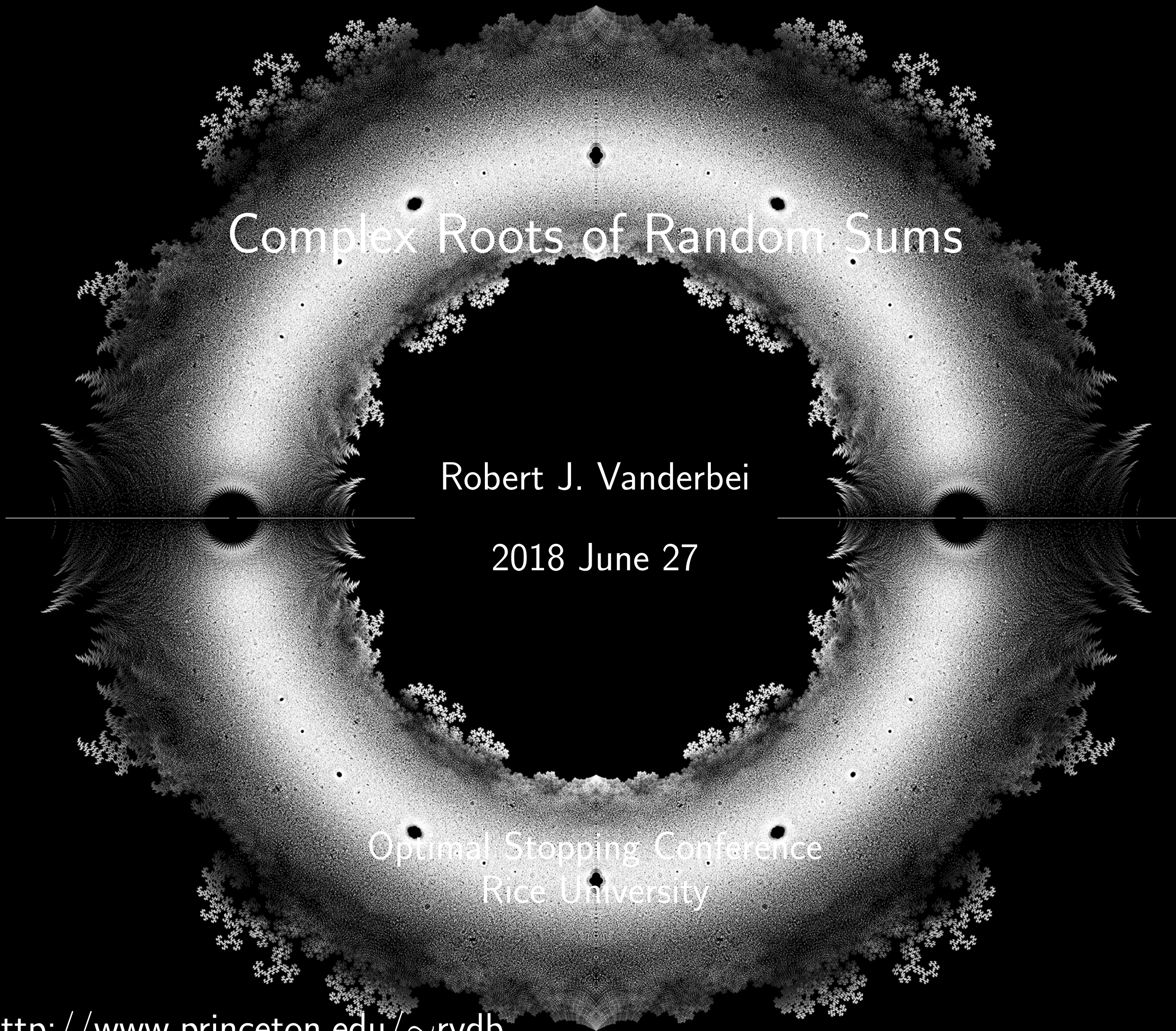

# Complex Roots of Random Sums

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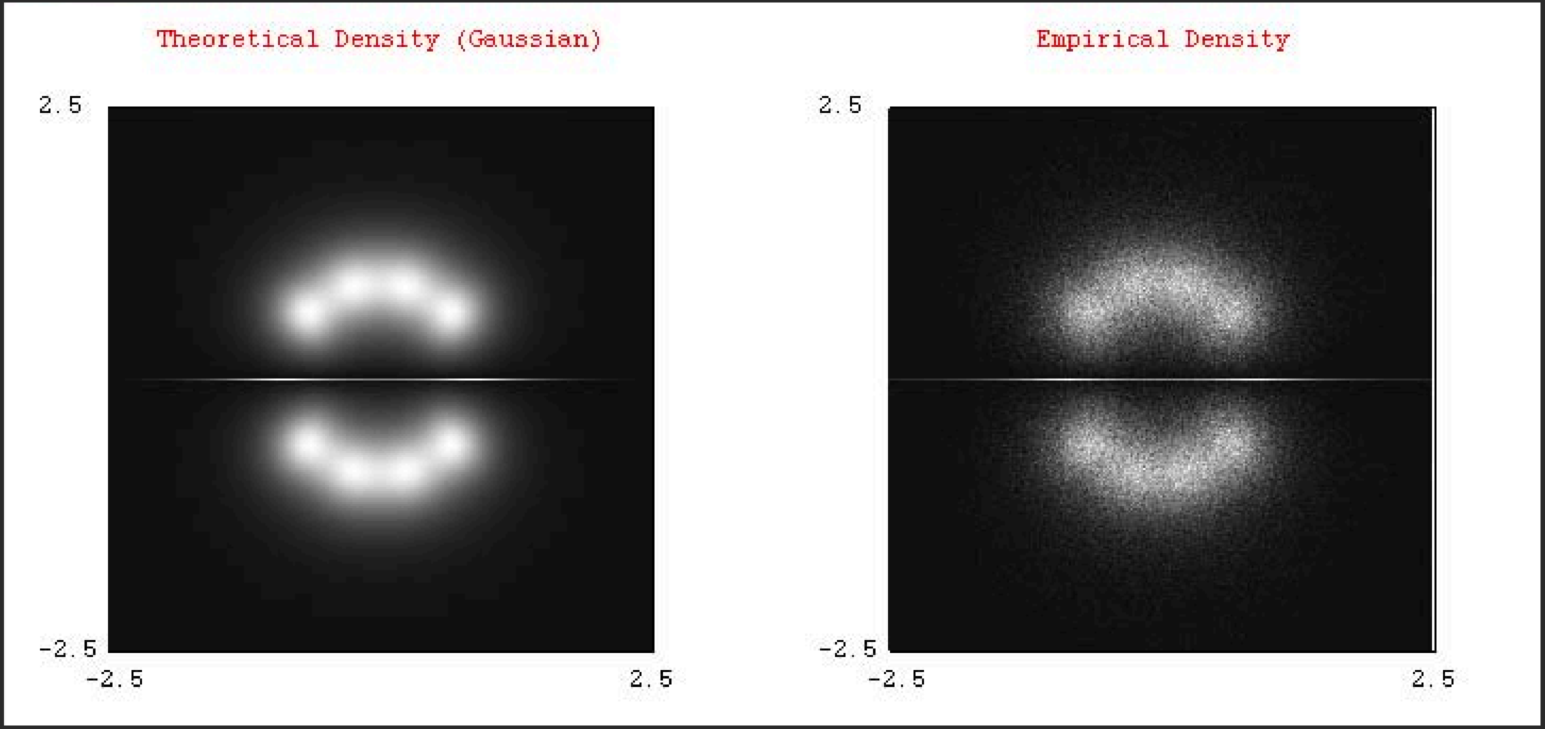
# Zeros of Random Polynomials (work with Larry Shepp)

Consider a random polynomial:

$$P_n(z) = \sum_{j=0}^{n} \eta_j z^j, \quad z \in \mathbb{C},$$

where $\eta_0, \ldots, \eta_n$ are independent standard normal random variables.
The distribution of the zeros for, say, $n = 5$ looks like this...



## Explicit Formula for any $n$:

*Let $\nu_n(\Omega)$ denote the number of zeros in a set $\Omega$ in the complex plane. For each measurable set $\Omega \subset \mathbb{C}$,*

$$\mathbf{E}\nu_n(\Omega) = \int_{\Omega} h_n(x+iy)dxdy + \int_{\Omega\cap\mathbb{R}} g_n(x)dx,$$

*where*

$$h_n = \frac{B_2 D_0^2 - B_0(B_1^2 + |A_1|^2) + B_1(A_0\overline{A_1} + \overline{A_0}A_1)}{\pi|z|^2 D_0^3},$$

*and*

$$g_n = \frac{(B_0 B_2 - B_1^2)^{1/2}}{\pi|z|B_0}$$

*and where*

$$B_k(z) = \sum_{j=0}^{n} j^k |z|^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1, 2,$$

$$A_k(z) = \sum_{j=0}^{n} j^k\, z^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1,$$

*and*

$$D_0(z) = \sqrt{B_0^2(z) - |A_0|^2(z)}.$$

# Explicit Formula for Limiting Density

*Limiting density in the complex plane:*

$$h = \lim_{n\to\infty} h_n = \frac{|B|D}{\pi}.$$

*Limiting density on the real axis:*

$$g = \lim_{n\to\infty} g_n = \frac{1}{\pi}|B|.$$

*Where*

$$B(z) = \frac{1}{1-|z|^2}, \quad z \in \mathbb{C},$$

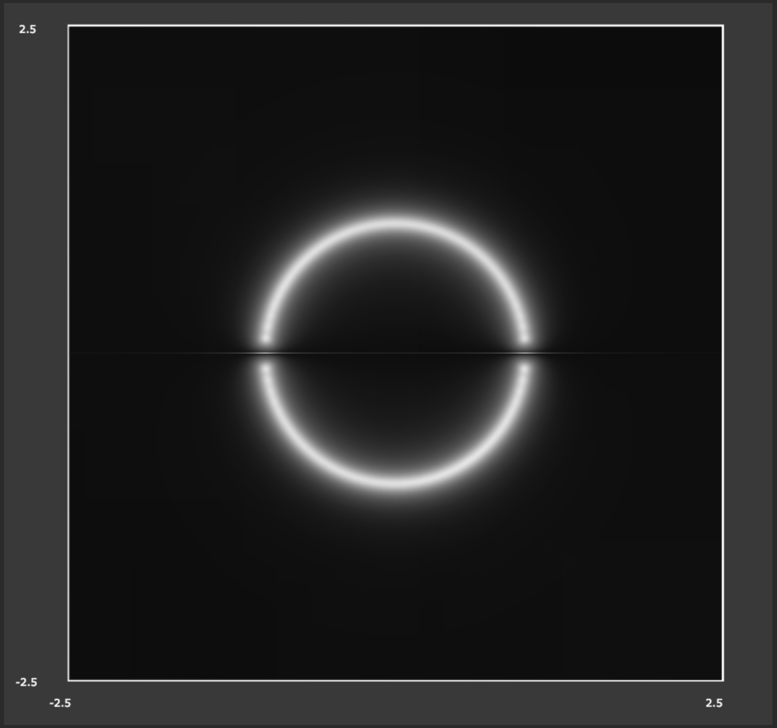
$$A(z) = \frac{1}{1-z^2}, \quad z \in \mathbb{C},$$

*and*

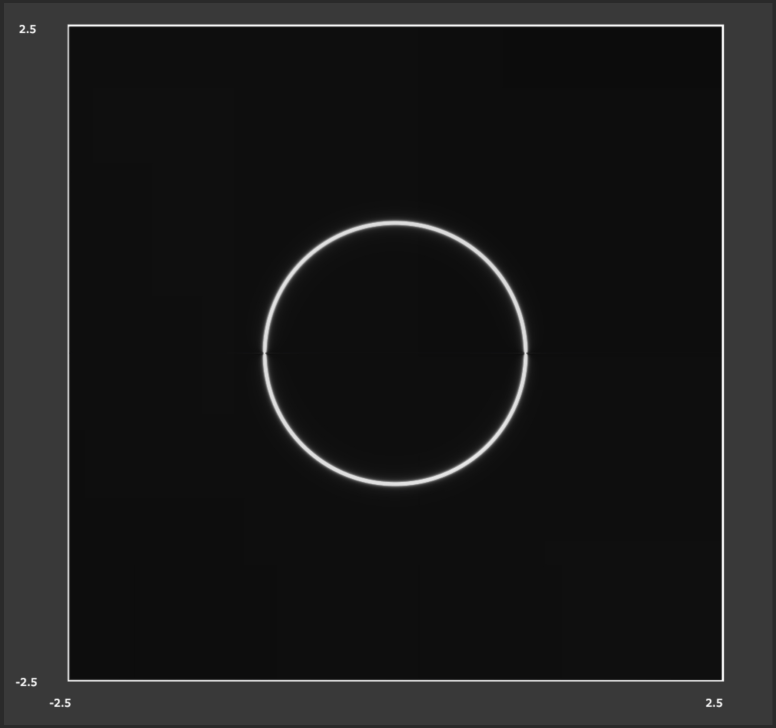
$$D(z) = \sqrt{B^2(z) - |A|^2(z)}.$$

# Large $n$ and the Limit

$n = 36$




$n = \infty$



# Nonlinear Stretches $n = \infty$

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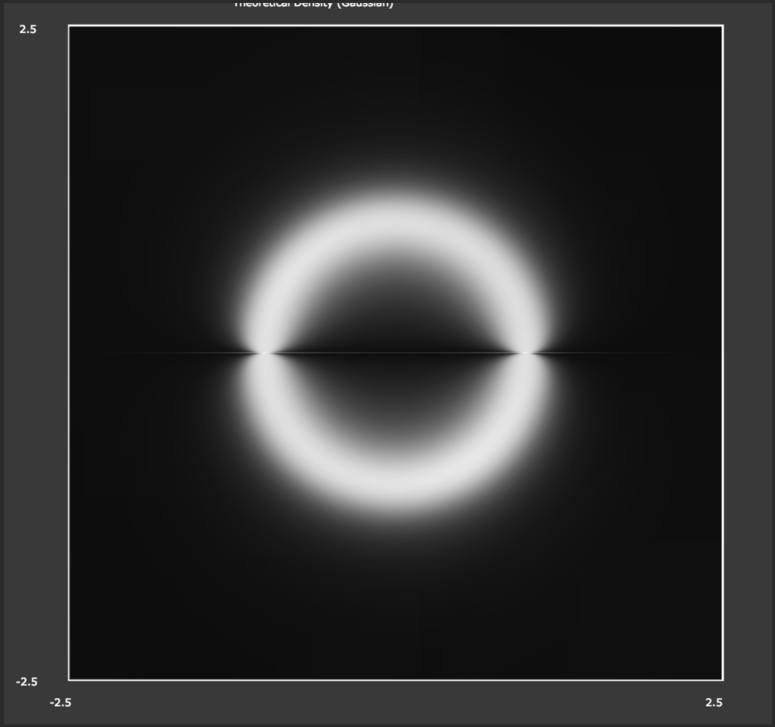


# New Work: Zeros of Random Sums

Consider a random sum of basis functions $f_j(z)$:

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z), \quad z \in \mathbb{C}$$

We assume that the basis functions are real on the real line.

Some interesting choices include:

| | | |
|---|---|---|
| Polynomials: | $f_j(z) = z^j$ | (okay, not so interesting) |
| Taylor Polynomials: | $f_j(z) = \dfrac{z^j}{j!}$ | |
| Weyl Polynomials: | $f_j(z) = \dfrac{z^j}{\sqrt{j!}}$ | |
| Root-Binomial Polynomials: | $f_j(z) = \sqrt{\binom{n}{j}}\, z^j$ | |
| Fourier Cosine Series: | $f_j(z) = \cos(jz)$ | |
| Fourier Sine/Cosine Series: | $f_j(z) = \begin{cases} \cos(\frac{j}{2}z) & j \text{ even} \\ \sin(\frac{j+1}{2}z) & j \text{ odd} \end{cases}$ | |

## Theorem 1

*For each measurable set* $\Omega \subset \{z \in \mathbb{C} \mid D_0(z) \neq 0\}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x+iy)dxdy,$$

*where*

$$h_n = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3},$$

*and where*

$$\begin{aligned} A_0(z) &= \sum_{j=0}^n f_j(z)^2, & B_0(z) &= \sum_{j=0}^n |f_j(z)|^2, \\ A_1(z) &= \sum_{j=0}^n f_j(z) f_j'(z), & B_1(z) &= \sum_{j=0}^n \overline{f_j(z)} f_j'(z), \\ A_2(z) &= \sum_{j=0}^n f_j'(z)^2, & B_2(z) &= \sum_{j=0}^n |f_j'(z)|^2, \end{aligned}$$

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}.$$

# Comparison to Polynomial Result

*For each measurable set* $\Omega \subset \{z = x + iy \in \mathbb{C} \mid y \neq 0\}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x+iy)\,dx\,dy,$$

*where*

$$h_n = \frac{B_2 D_0^2 - B_0(B_1^2 + |A_1|^2) + B_1(A_0\overline{A_1} + \overline{A_0}A_1)}{\pi |z|^2 D_0^3},$$

*and where*

$$A_0(z) = \sum_{j=0}^{n} z^{2j}, \qquad B_0(z) = \sum_{j=0}^{n} |z|^{2j},$$

$$A_1(z) = \sum_{j=0}^{n} j z^{2j}, \qquad B_1(z) = \sum_{j=0}^{n} j|z|^{2j},$$

$$A_2(z) = \sum_{j=0}^{n} j^2 z^{2j}, \qquad B_2(z) = \sum_{j=0}^{n} j^2|z|^{2j},$$

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}.$$

## Theorem 2

*On the real line, the function $D_0$ vanishes. For each Borel measurable set $\Omega \subset \mathbb{R}$,*

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega g_n(x)dx,$$

*where*

$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}.}{\pi B_0(x)}$$

## Main Idea

From the *argument principle*, we get

$$\nu_n(\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)} dz.$$

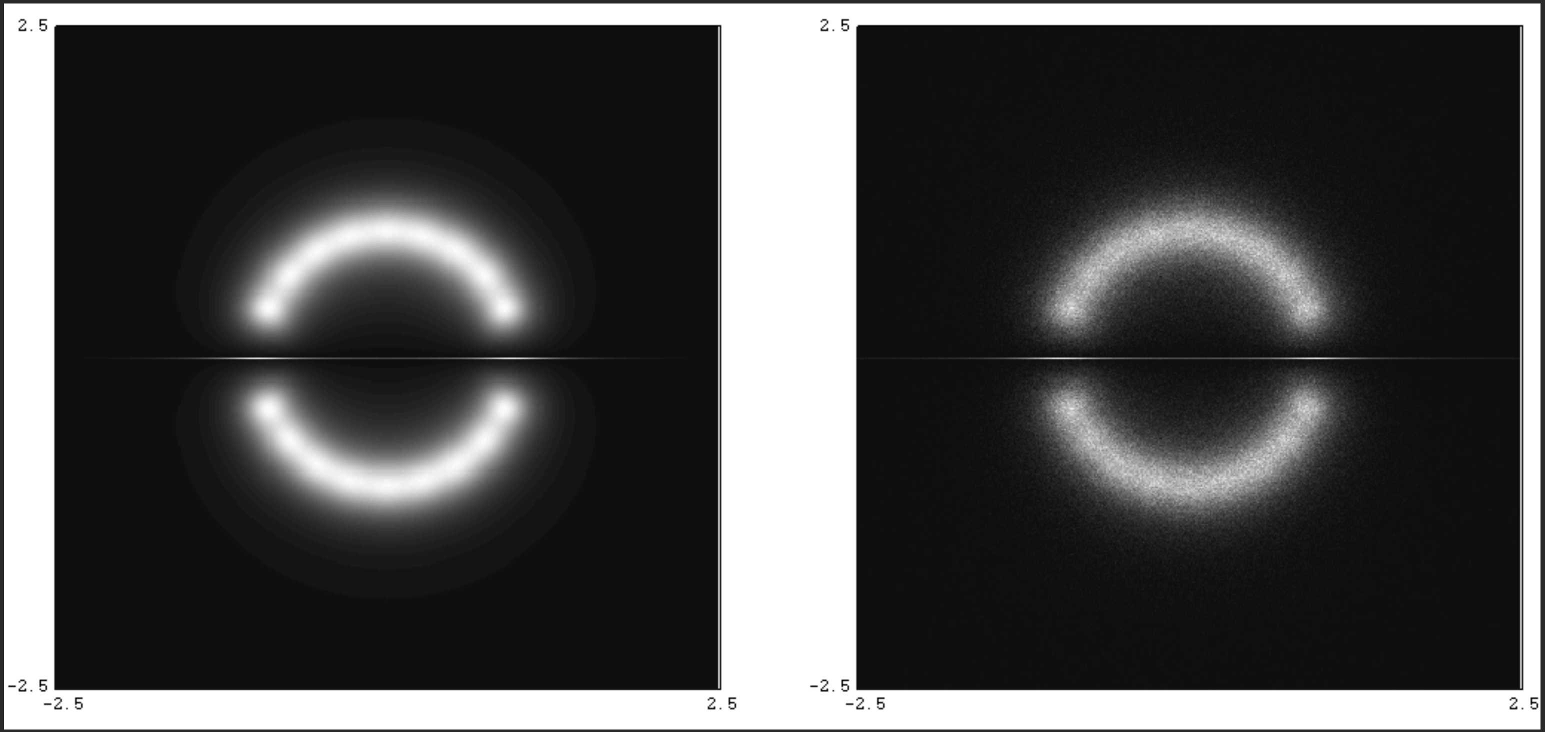
Taking the expectation, interchanging expectation and integration and then applying *Stoke's Theorem*, we get

$$\begin{aligned}\mathbf{E}\nu_n(\Omega) &= \frac{1}{2\pi i}\int_{\partial\Omega} \mathbf{E}\frac{P_n'(z)}{P_n(z)} dz \\ &= \frac{1}{\pi}\int_{\Omega} \frac{\partial}{\partial \overline{z}}\mathbf{E}\frac{P_n'(z)}{P_n(z)} dx dy\end{aligned}$$

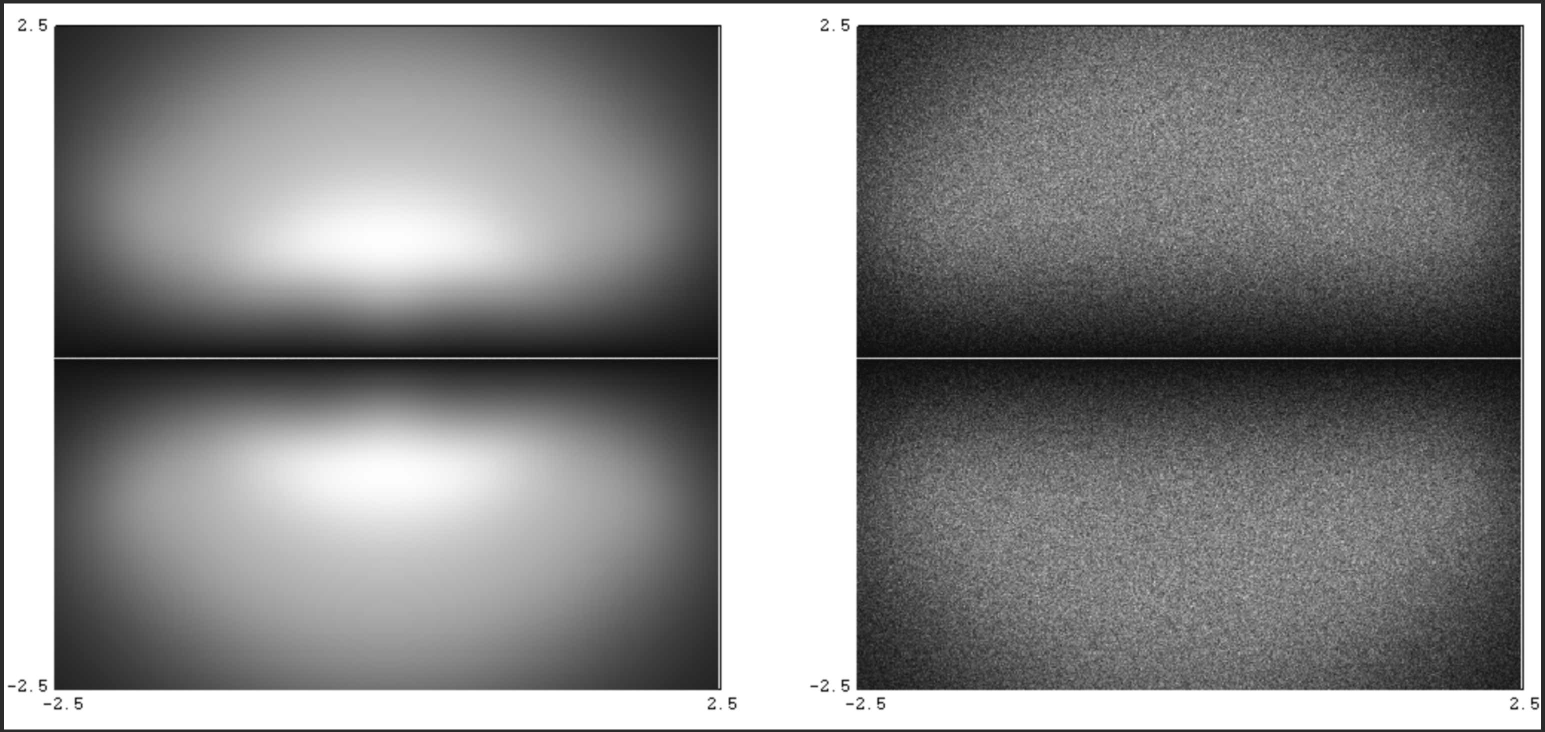
The rest is tedious (and perhaps nontrivial) algebra/calculus.

NOTE: The derivative of the expectation of $P_n'(z)/P_n(z)$ is a function of both $z$ and $\overline{z}$.

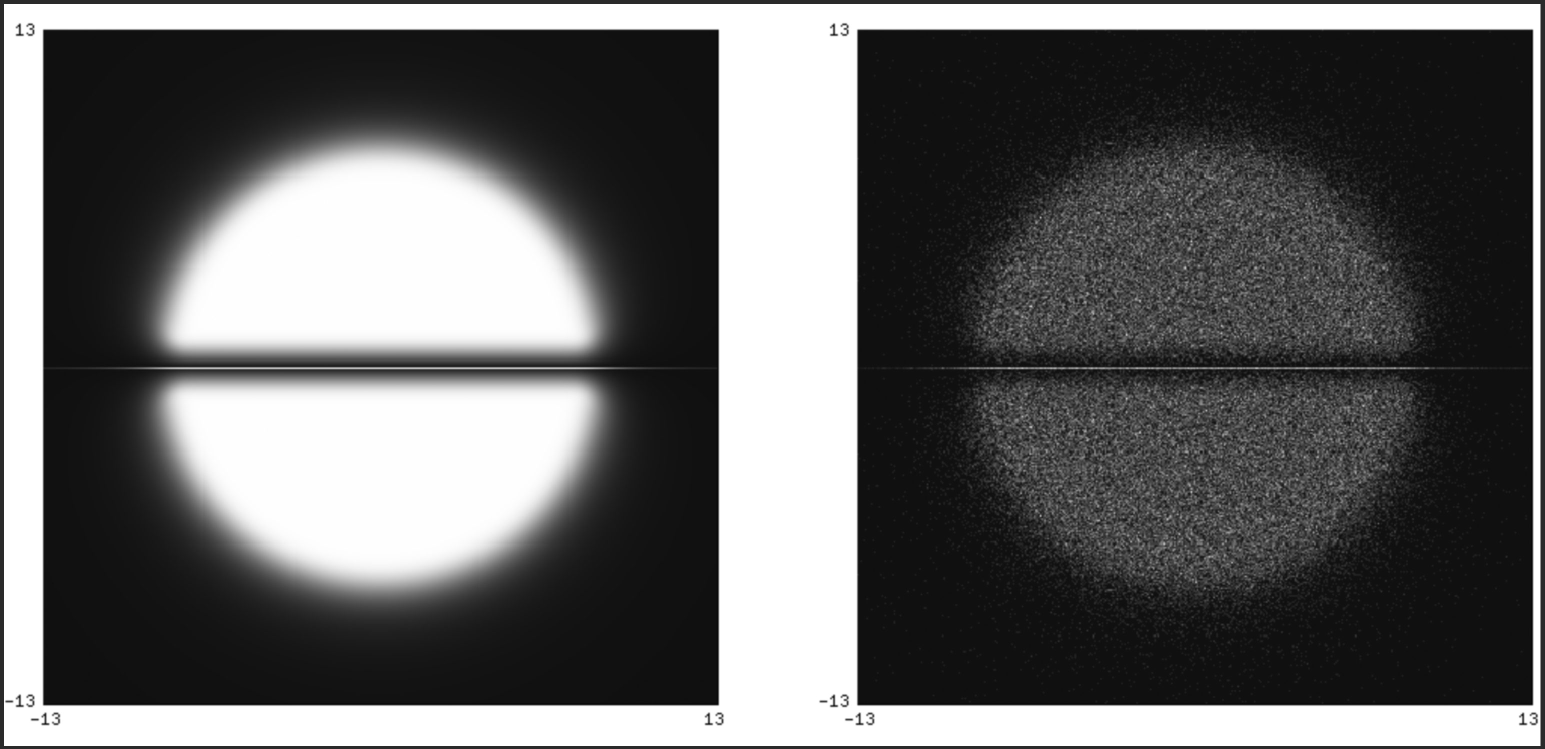
## **Polynomials:** $f_j(z) = z^j, \quad n = 10$



# Weyl Polynomials: $f_j(z) = z^j/\sqrt{j!}$, $n = 10$



# Weyl Polynomials: $f_j(z) = z^j/\sqrt{j!}$, $n = 80$



# Limiting Case for $f_j(z) = z^j/\sqrt{j!}$ (Weyl Polynomials)

$$A_0(z) = \sum_{j=0}^{\infty} f_j(z)^2 = e^{z^2} \qquad B_0(z) = \sum_{j=0}^{\infty} |f_j(z)|^2 = e^{|z|^2}$$

$$A_1(z) = \sum_{j=0}^{\infty} f_j(z) f_j'(z) = z e^{z^2} \qquad B_1(z) = \sum_{j=0}^{\infty} \overline{f_j(z)} f_j'(z) = \overline{z} e^{|z|^2}$$

$$A_2(z) = \sum_{j=0}^{\infty} f_j'(z)^2 = (z^2+1) e^{z^2} \qquad B_2(z) = \sum_{j=0}^{\infty} |f_j'(z)|^2 = (|z|^2+1) e^{|z|^2}$$

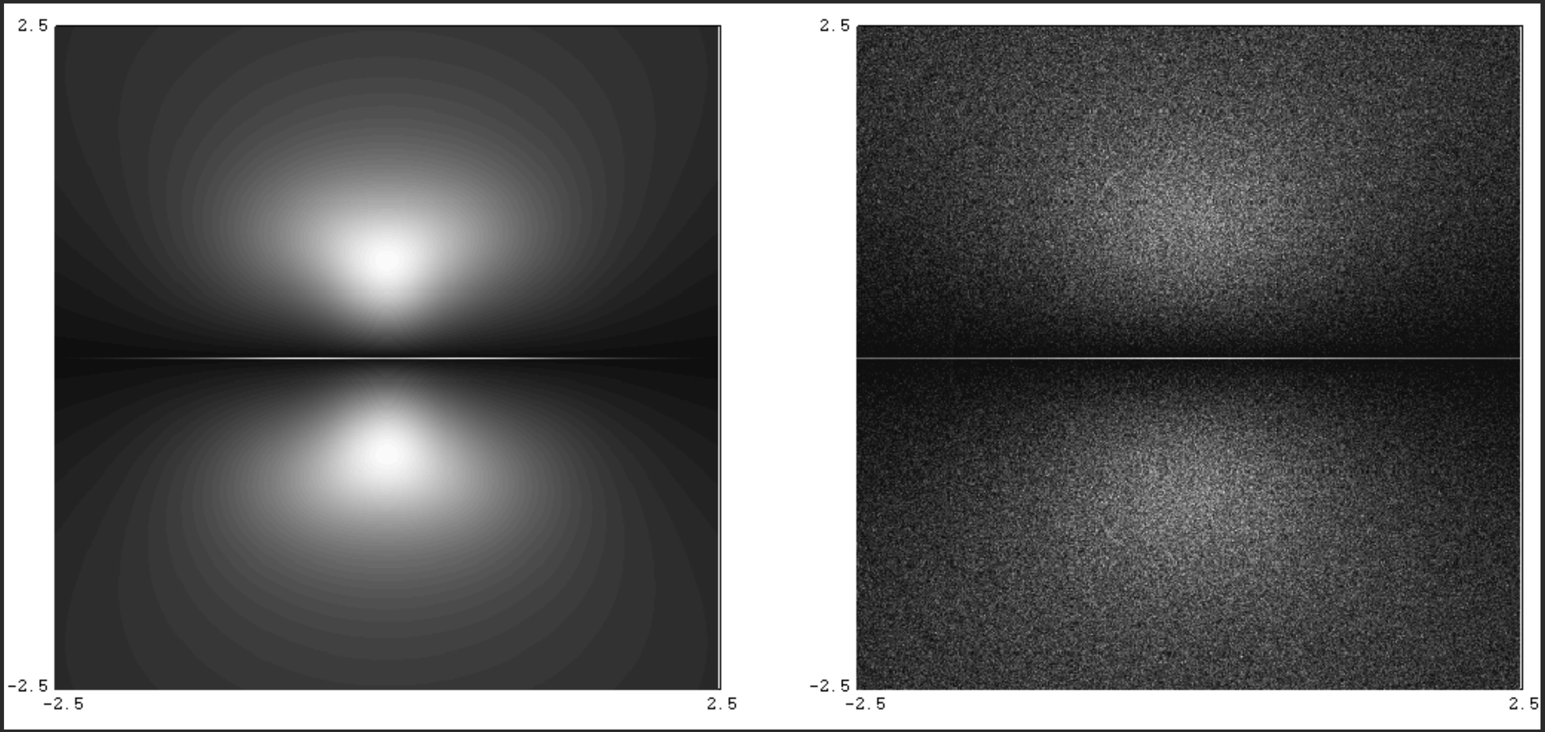
$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2} = \sqrt{e^{2|z|^2} - e^{z^2+\overline{z}^2}}$$

$$h(z) = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3} = \frac{\left(1 + ((z-\overline{z})^2 - 1) e^{(z-\overline{z})^2}\right)}{\pi \left(1 - e^{(z-\overline{z})^2}\right)^{3/2}}$$

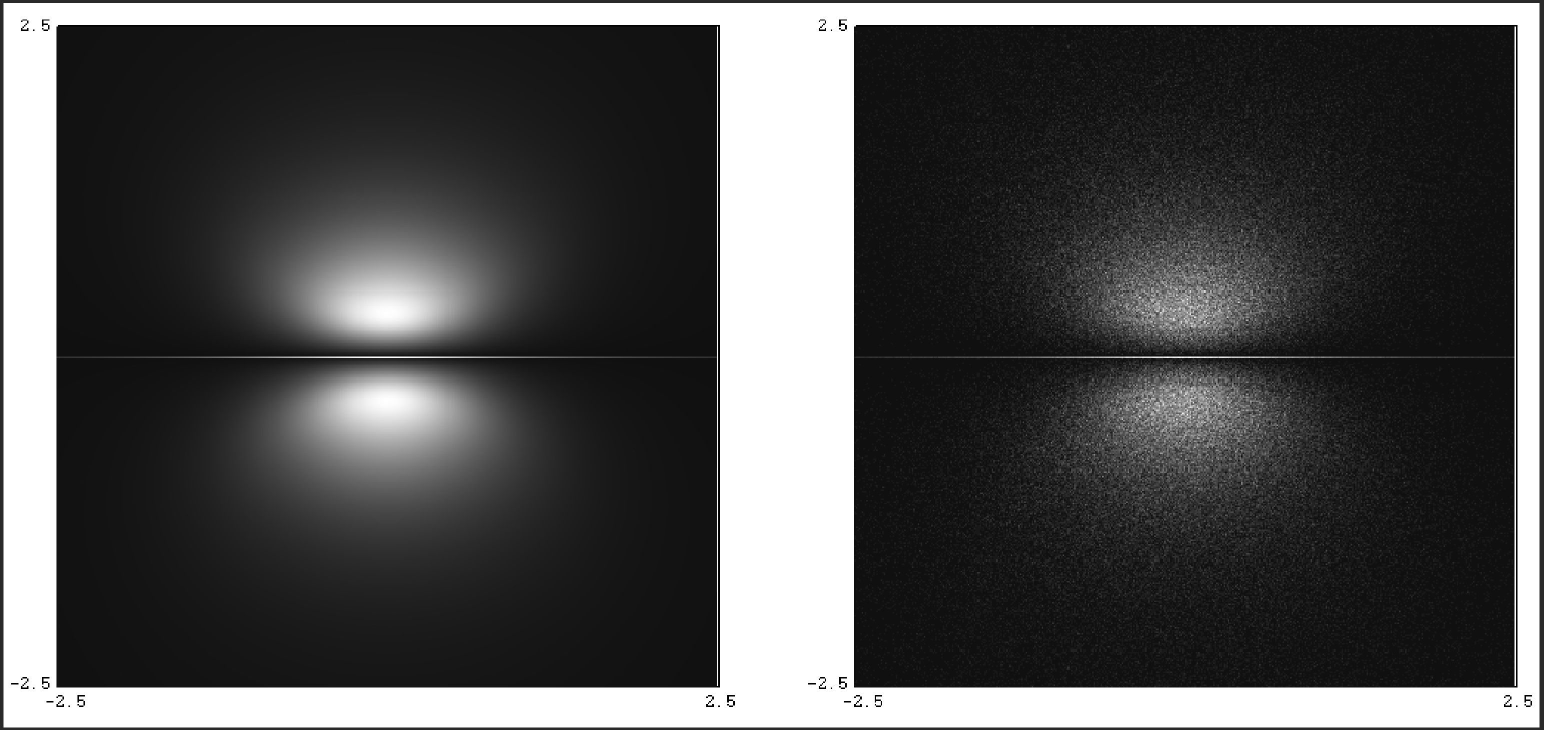
$$= \frac{\left(1 + (-4y^2 - 1) e^{-4y^2}\right)}{\pi \left(1 - e^{-4y^2}\right)^{3/2}}$$

$$g(x) = \frac{\sqrt{A_2(x) A_0(x) - A_1(x)^2}}{\pi B_0(x)} = \frac{1}{\pi}$$

# Taylor Polynomials: $f_j(z) = z^j/j!, \quad n = 10$



## Root Binomial Polynomials: $f_j(z) = \sqrt{\binom{n}{j}} z^j, \quad n = 10$



# Root-Binomial Polynomials: $f_j(z) = \sqrt{\binom{n}{j}} z^j$

$$A_0(z) = (1 + z^2)^n \qquad B_0(z) = (1 + |z|^2)^n$$

$$A_1(z) = nz(1 + z^2)^{n-1} \qquad B_1(z) = n\overline{z}(1 + |z|^2)^{n-1}$$

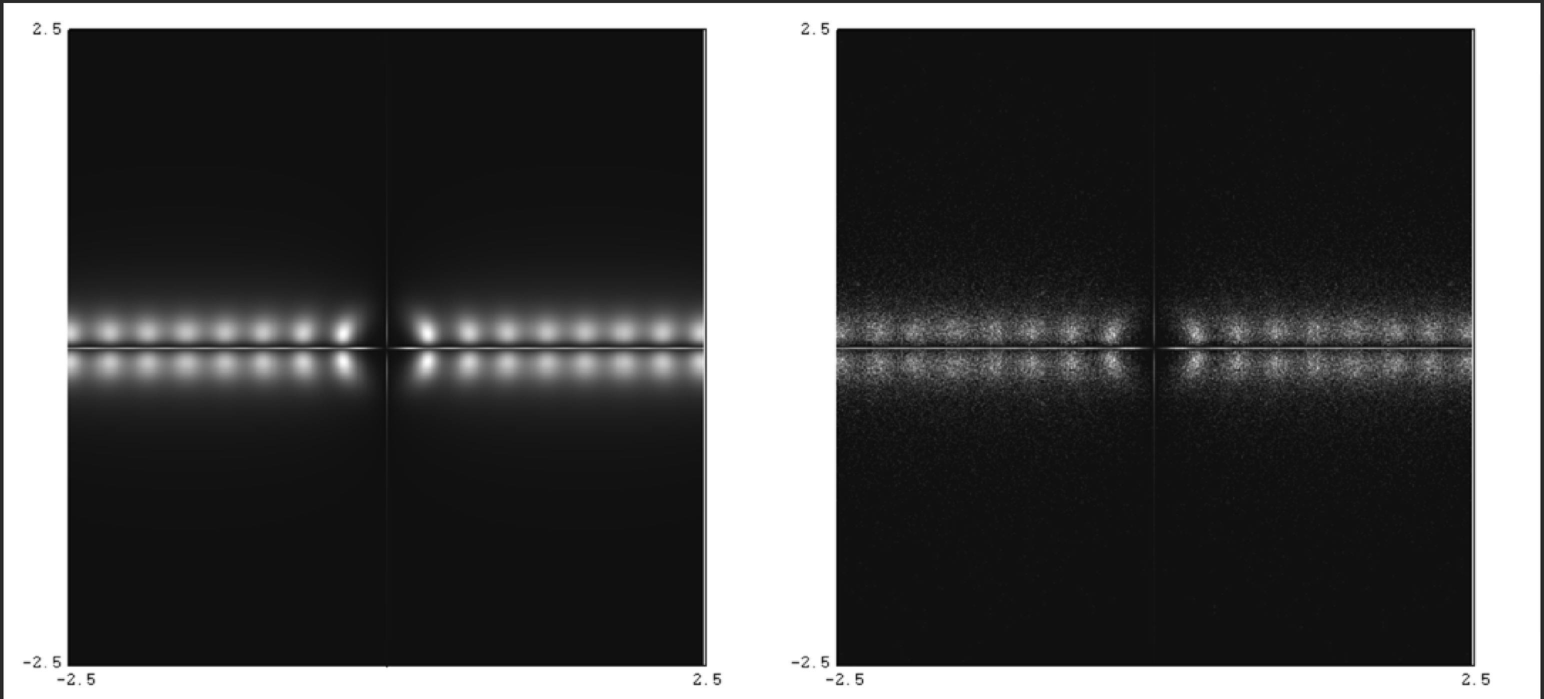
$$A_2(z) = n(1 + nz^2)(1 + z^2)^{n-2} \qquad B_2(z) = n(1 + n|z|^2)(1 + |z|^2)^{n-2}$$

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2} = \sqrt{(1 + |z|^2)^{2n} - |1 + z^2|^{2n}}$$

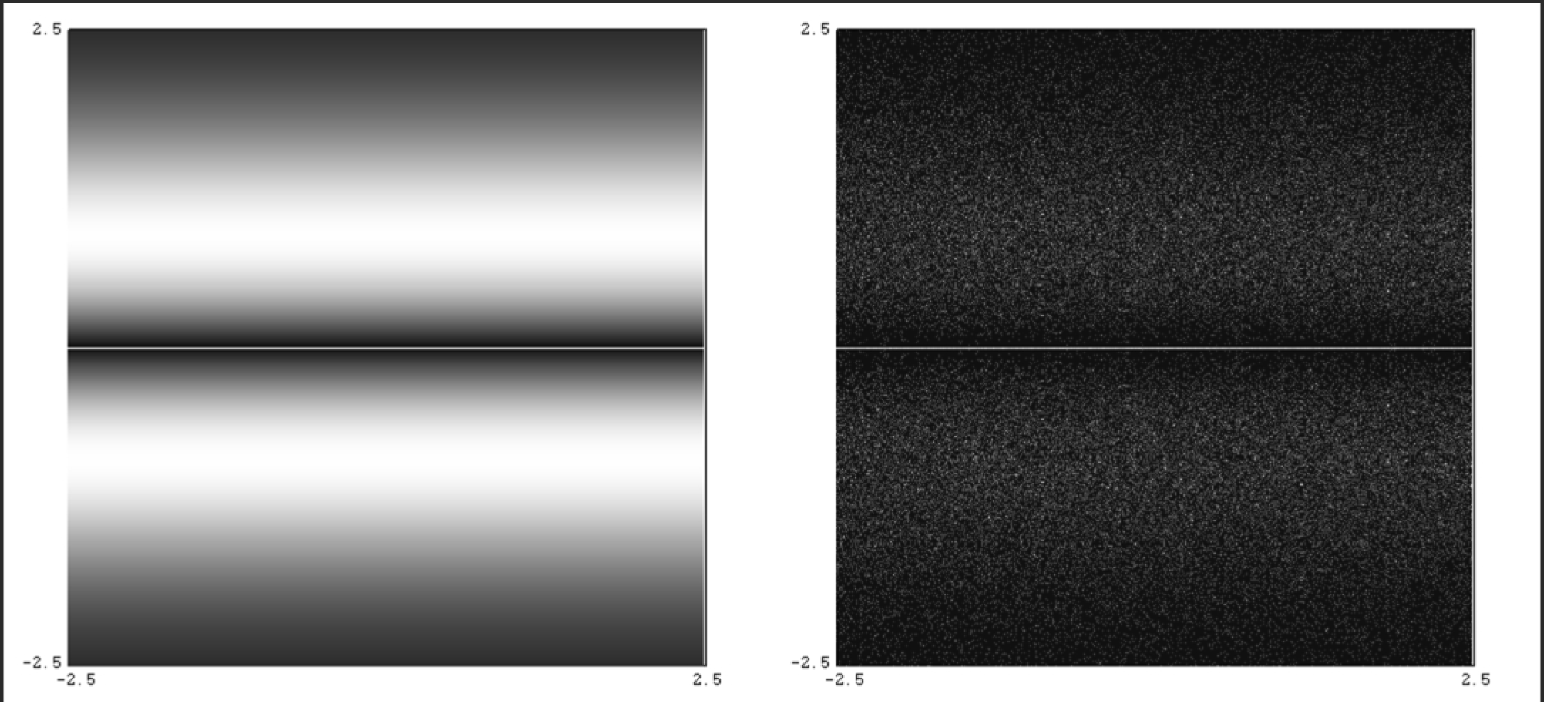
$$h_n = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3} = ???$$

$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}}{\pi B_0(x)} = \frac{\sqrt{n}}{\pi} \frac{1}{1 + x^2} \qquad \longleftarrow \qquad \text{Cauchy distribution}$$

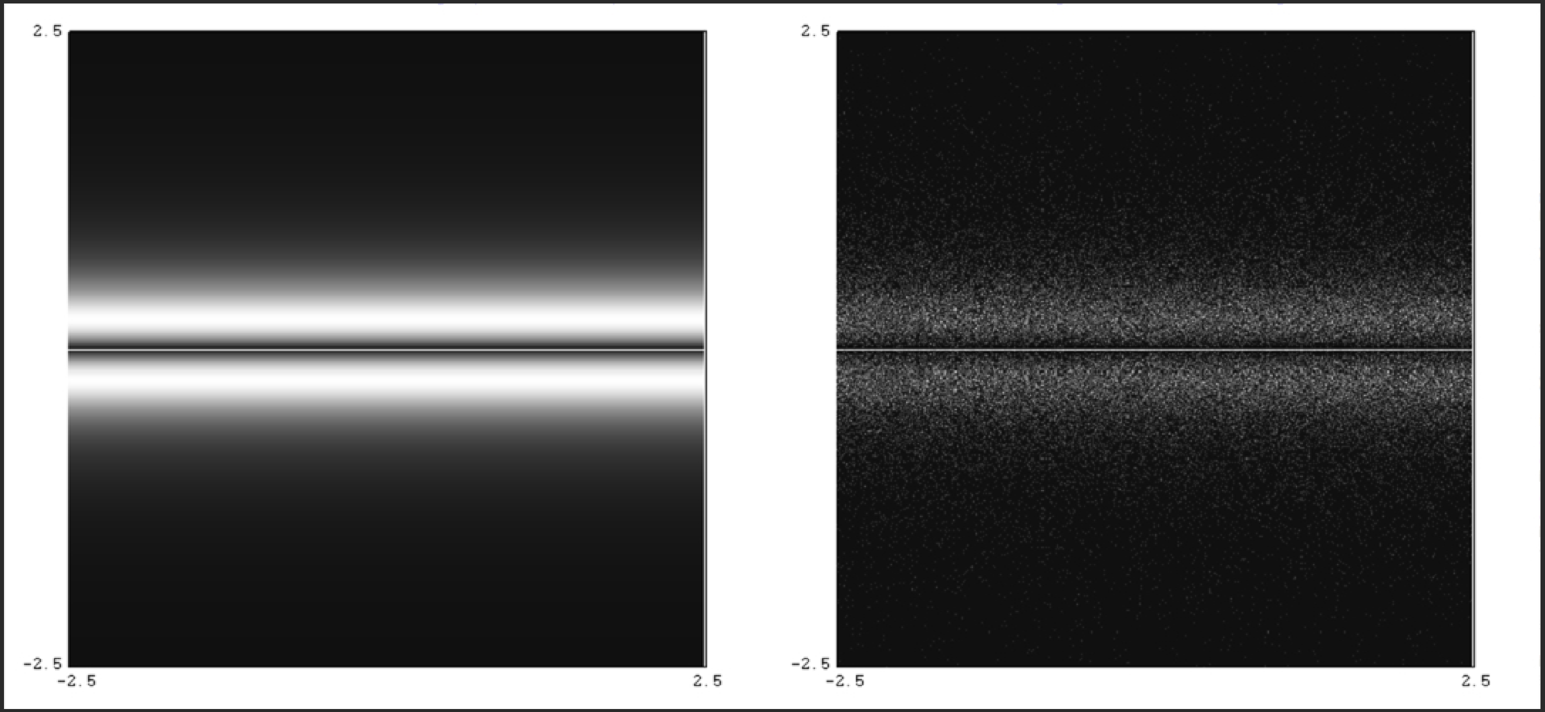
**Fourier:** $\alpha_0 + \alpha_1 \cos(z) + \alpha_2 \cos(2z) + \cdots + \alpha_9 \cos(9z) + \alpha_{10} \cos(10z)$



# Sine/Cosine Fourier: $\alpha_0 + \alpha_1 \sin(z) + \alpha_2 \cos(z)$



$$\alpha_0 + \alpha_1 \sin(z) + \alpha_2 \cos(z) + \cdots + \alpha_9 \sin(5z) + \alpha_{10} \cos(5z)$$



# Fourier Sine/Cosine Series $(n = 2m)$

$$A_0(z) = m + 1 \qquad B_0(z) = m + 1 + 2\sum_{j=1}^{m} \sinh^2(jy)$$

$$A_1(z) = 0 \qquad B_1(z) = -i\sum_{j=1}^{m} j\cosh(jy)\sinh(jy)$$

$$A_2(z) = m(m+1)(2m+1)/6 \qquad B_2(z) = 2\sum_{j=1}^{m} j^2\sinh^2(jy)$$

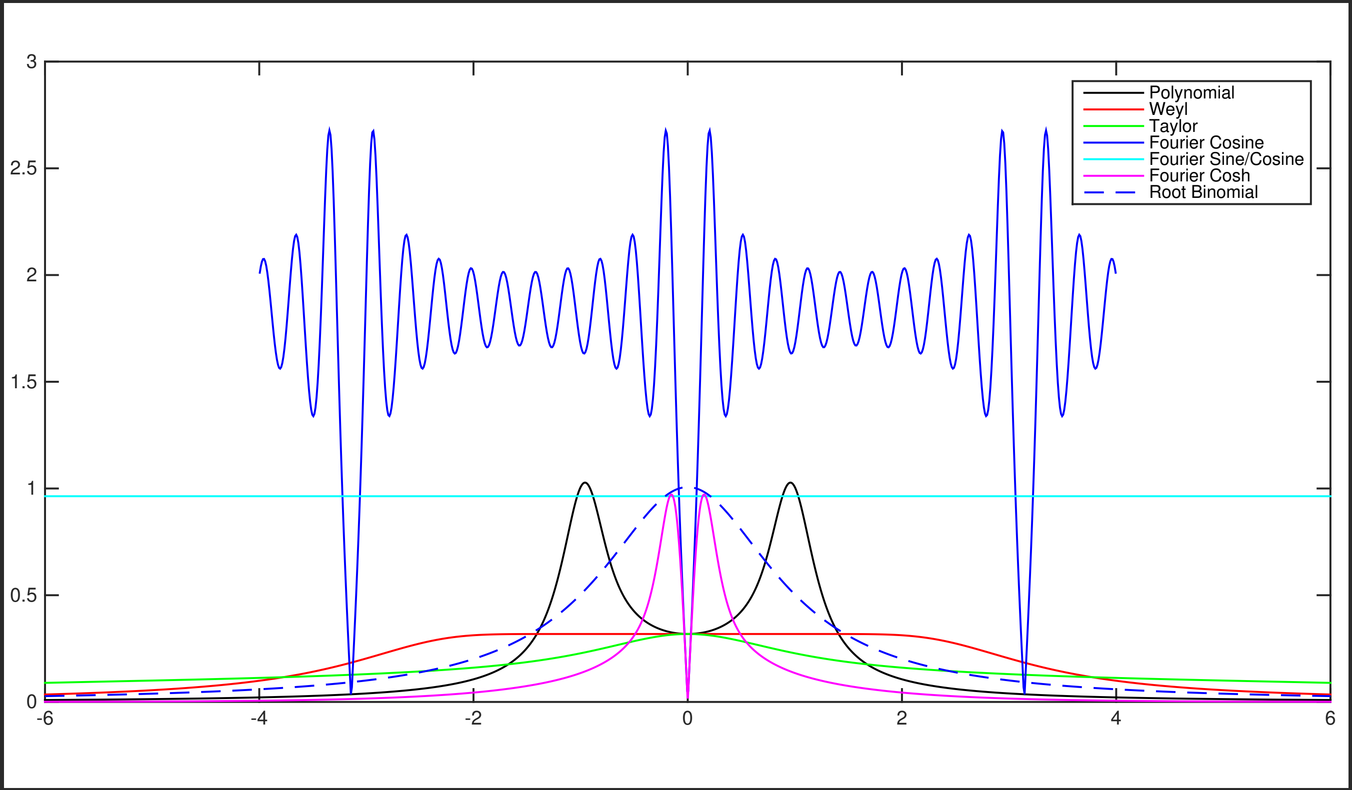
$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}$$

$$h_n = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3} = \frac{B_2 D_0^2 - B_0|B_1|^2}{\pi D_0^3}$$

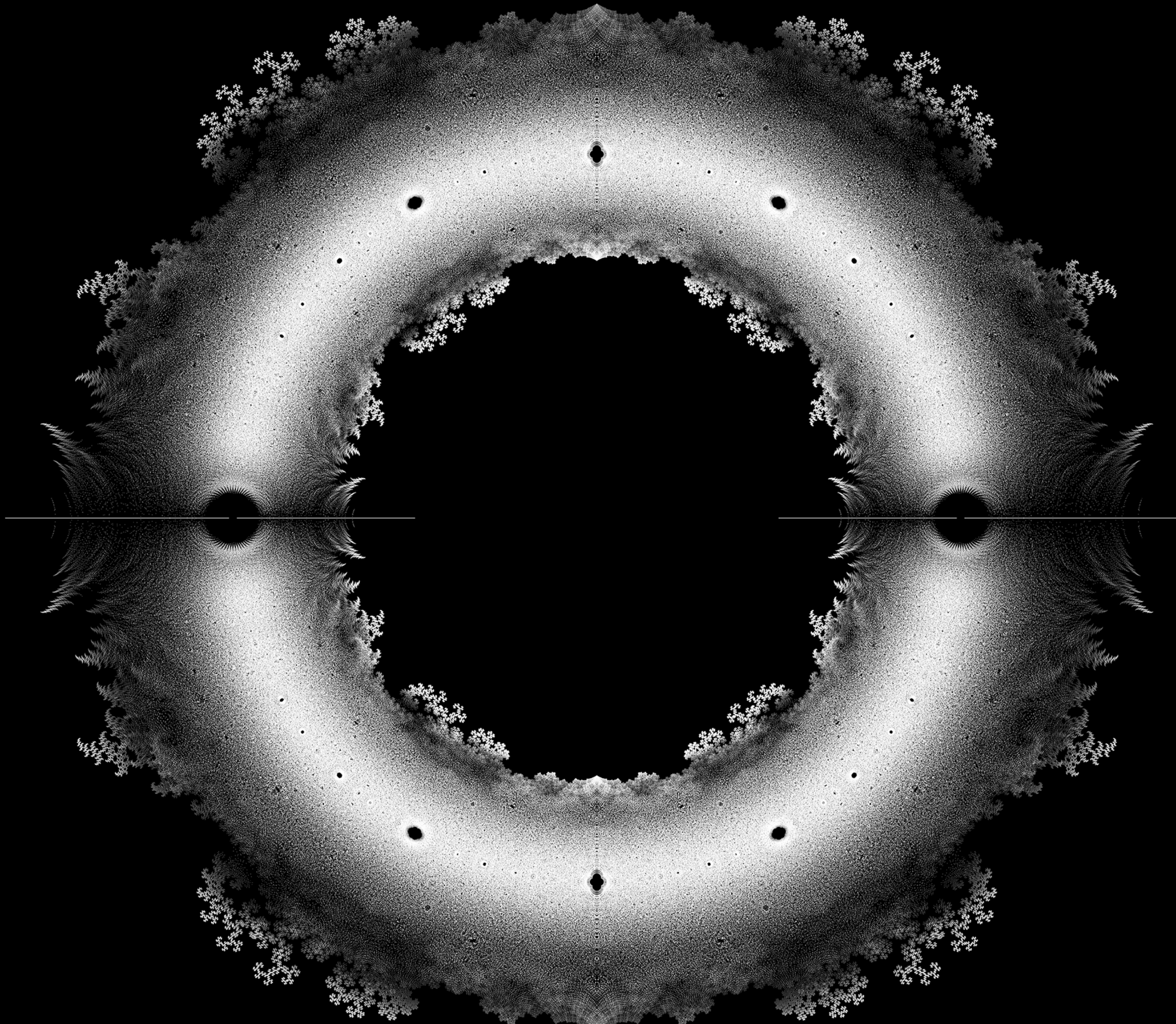
$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}}{\pi B_0(x)}$$

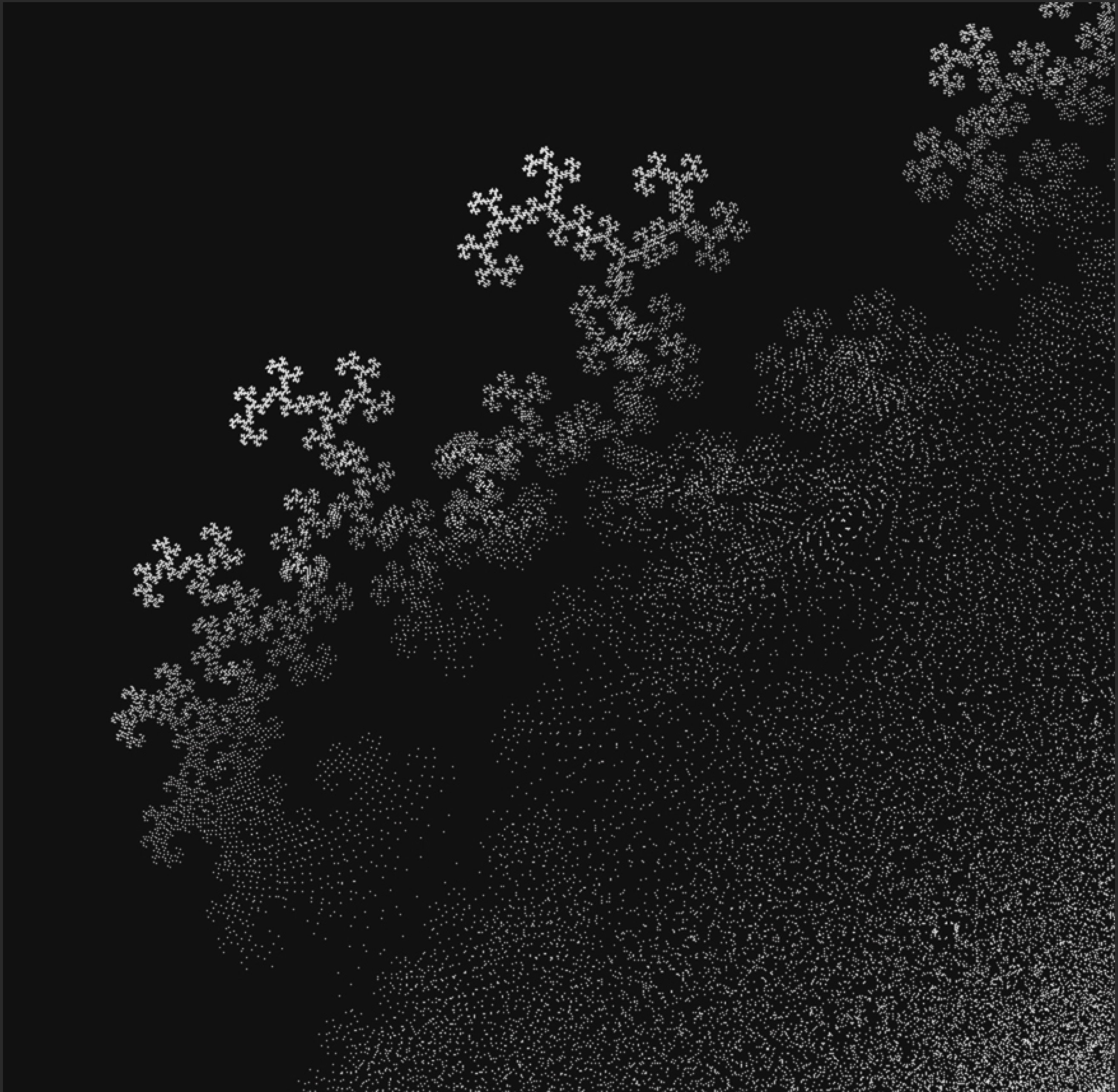
# Densities of the Real Roots $n = 10$

The function $g_n$ for $n = 10$ for several choices of the $f_j$'s:



## Polynomial Roots $\eta_j = \pm 1$ $n = 18$





# References


- *The Complex Zeros of Random Polynomials*,
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Thank You!