

A New Flat Map of a Sphere via Stress Minimization

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<https://vanderbei.princeton.edu>

Best Inventions of 2021

TIME

A SAVVIER WAY TO COMMUTE



A MORE ACCURATE WORLD MAP



A DIY ROBOTIC ARM

100 Amazing Innovations Changing How We Live

A SLEEKER MOUTHWASH



A SIMPLER WAY TO FOCUS

A SMARTER SLIP-ON RUNNING SHOE

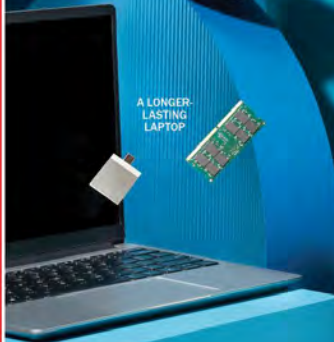


A TOY THAT KEEPS KIDS CALM



A MINI-CAR THAT TEACHES CODING

A LONGER-LASTING LAPTOP



NEXT-GEN NOODLES

A More Accurate World Map

Gott-Goldberg-Vanderbei Projection



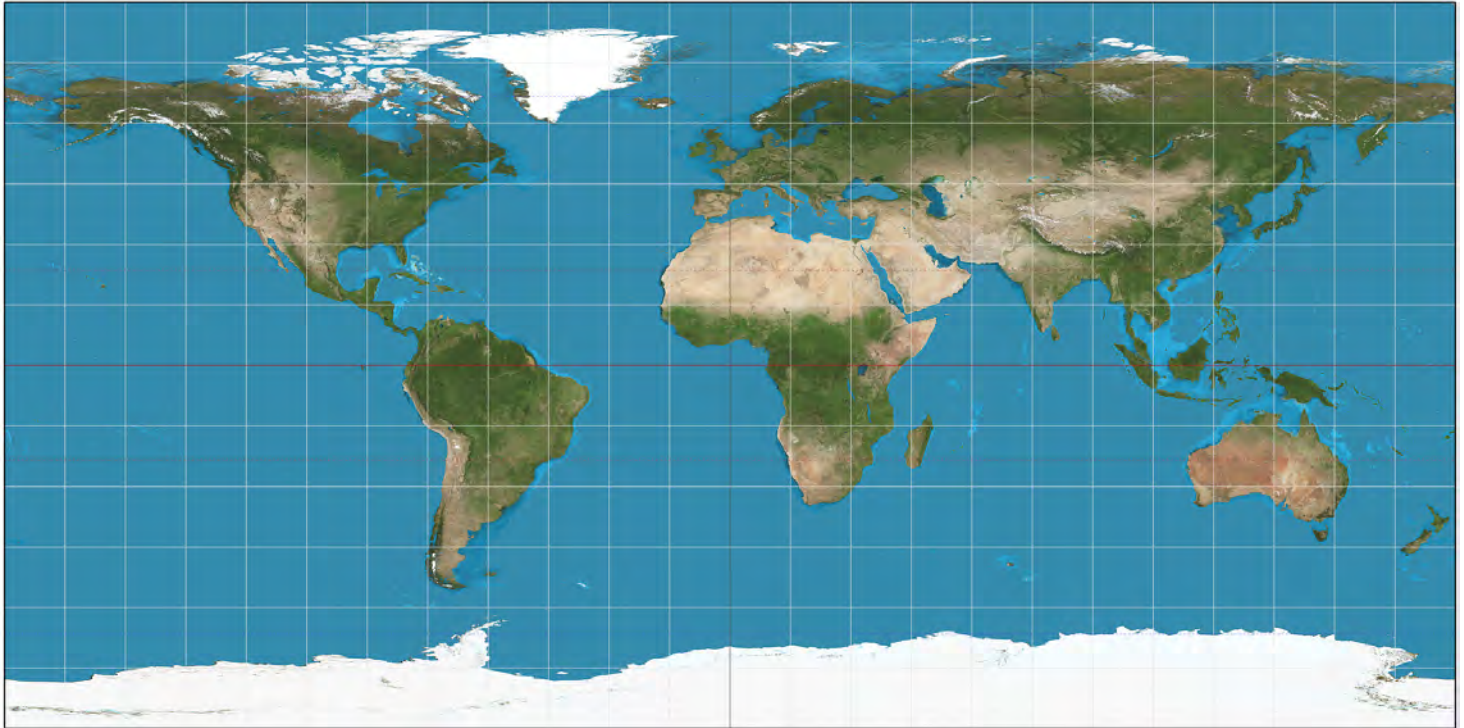
NOVEMBER 10, 2021 6:08 AM EST

In 1569, when Flemish cartographer Gerardus Mercator flattened the earth's cylindrical surface onto paper, he gave sailors the tools to navigate ocean voyages. But he also distorted the size of countries nearest the poles—North America appears abnormally large, for example. Despite the inaccuracies, the Mercator projection became the norm, and was even the basis of Google Maps until as recently as 2018. Astrophysicist J. Richard Gott, along with colleagues David Goldberg and Robert Vanderbei, set out in 2019 to fix the inaccuracies and came up with a double-sided map that is similar to a vinyl record in shape. It improves geographical parity between continents, finally representing the southern hemisphere as fairly as possible on paper. **The map** is free to access online, and the scientists are working with publishers to make it widely available for sale in the future. —*Eloise Barry*

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Classic Map Projections...

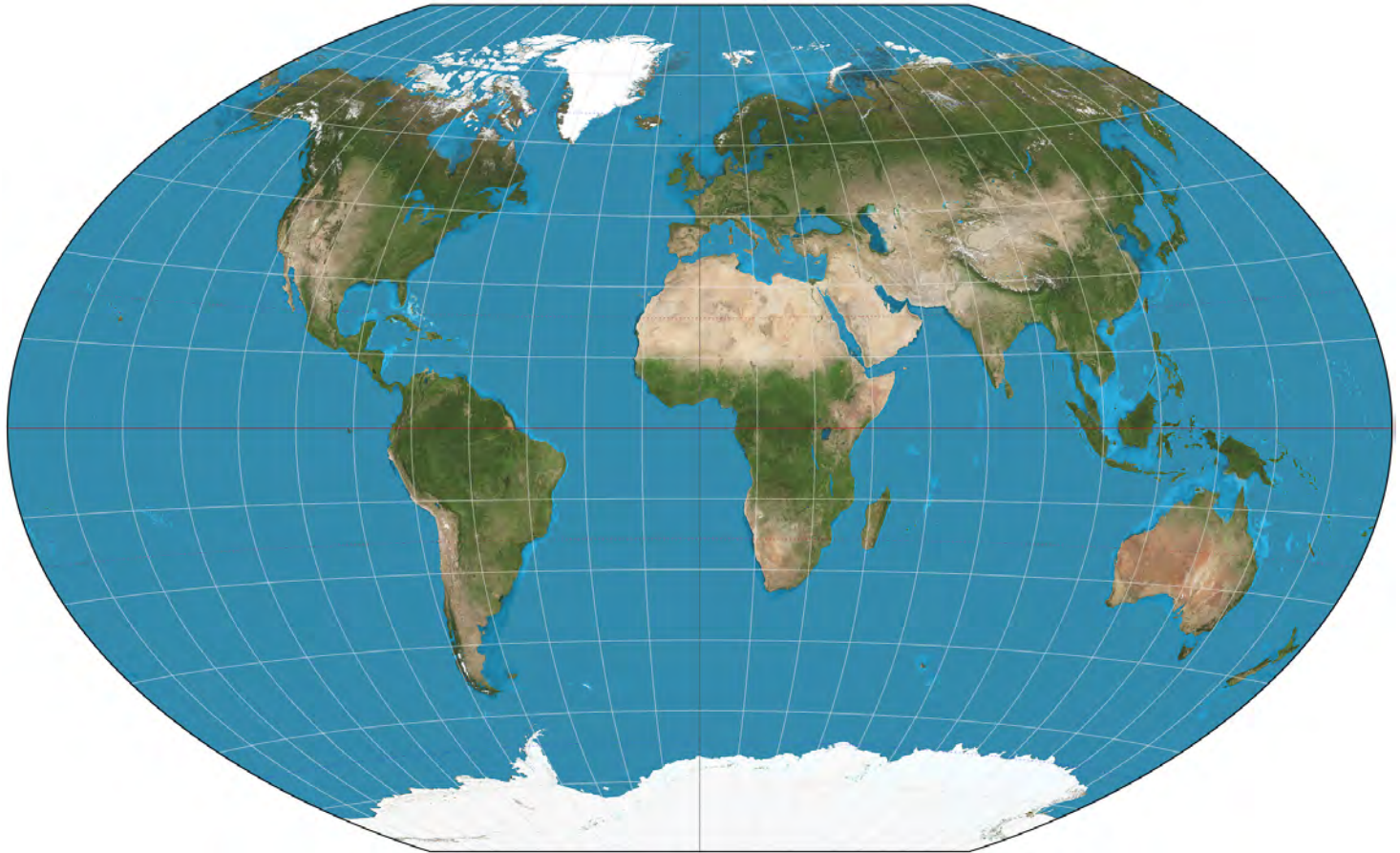
Equirectangular Map



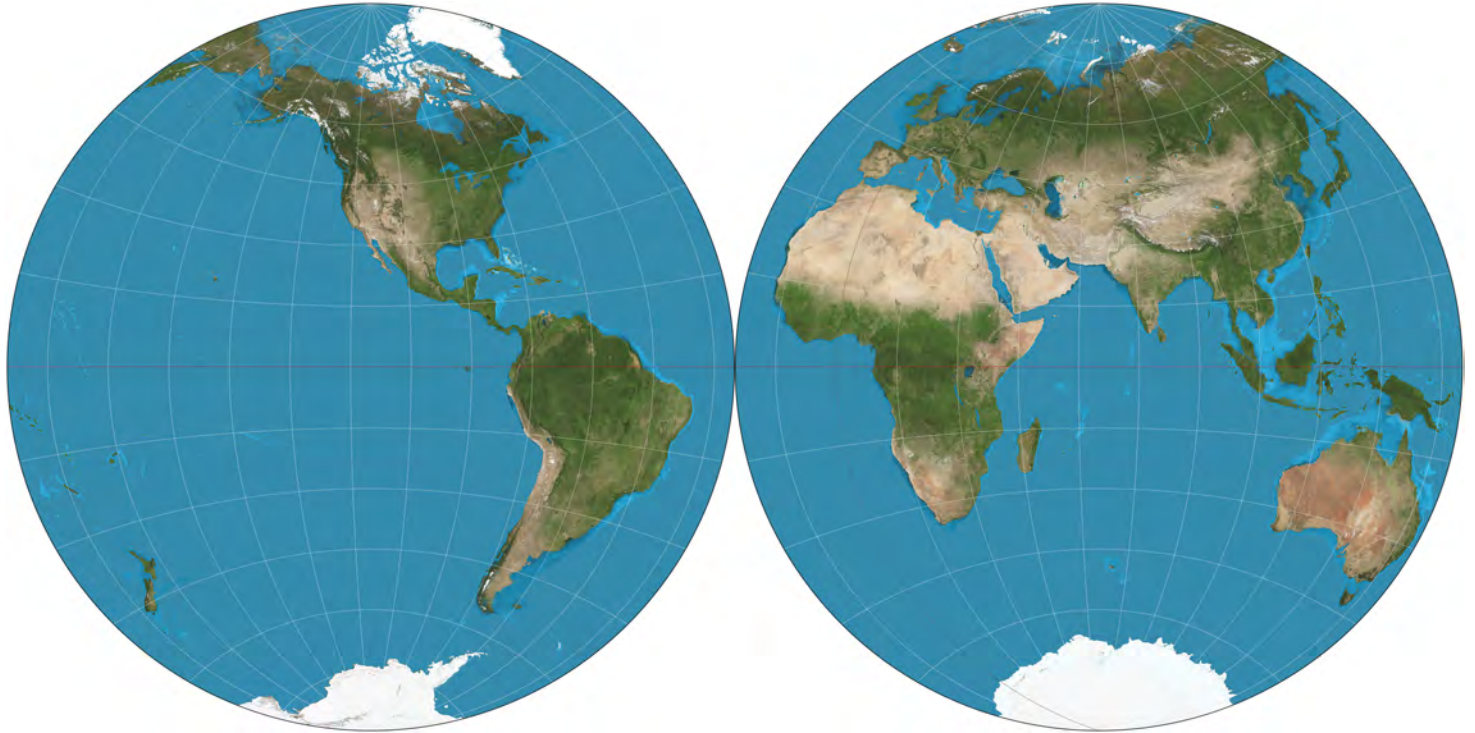
Mercator Map



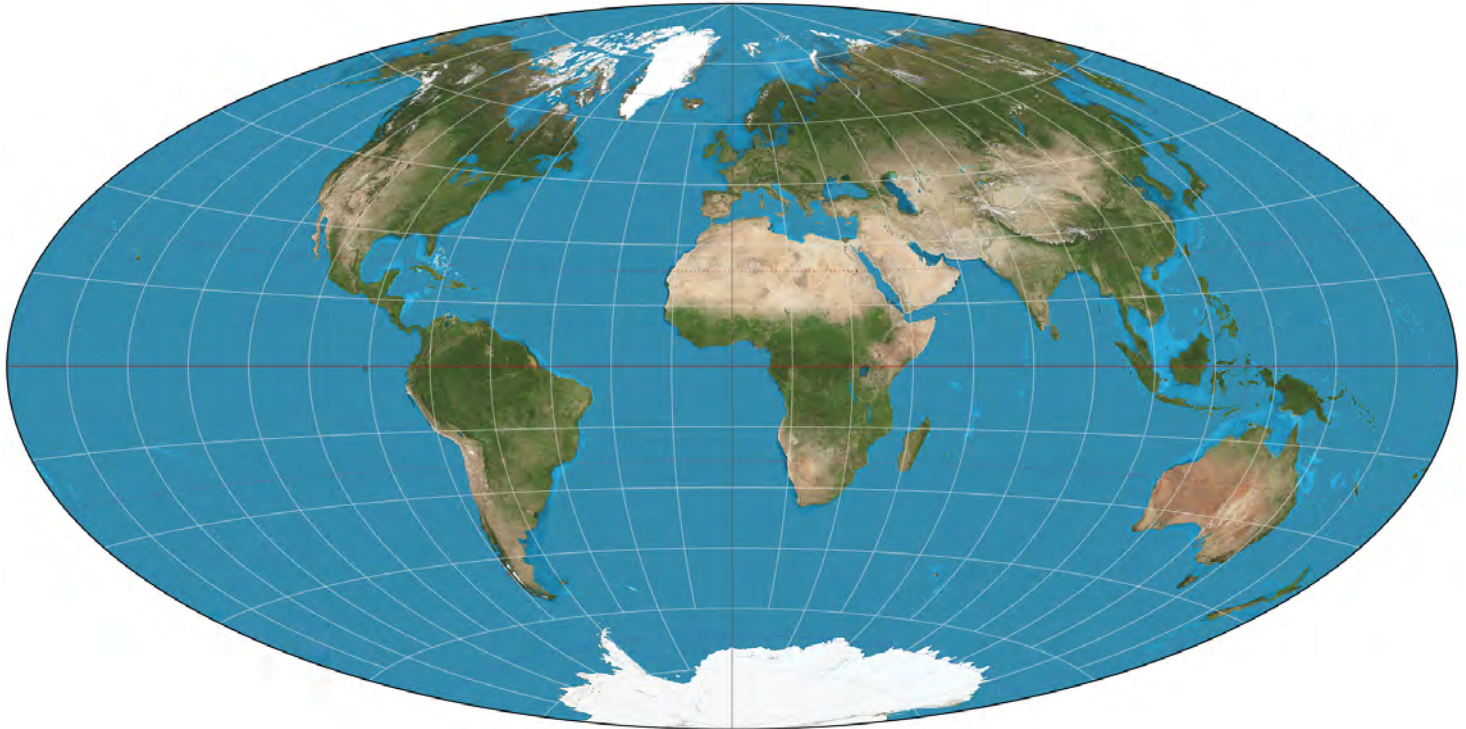
Winkel Tripel Map



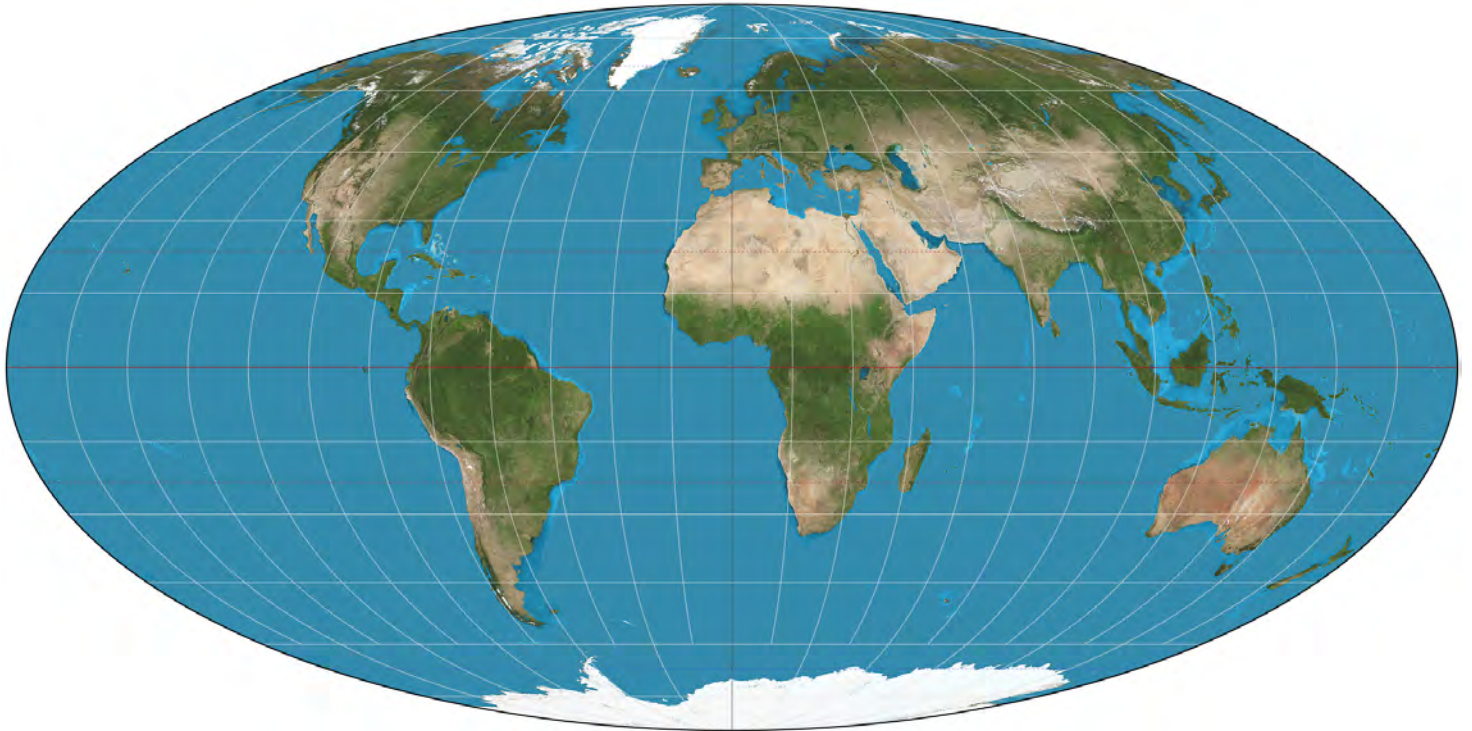
Nicolosi Map



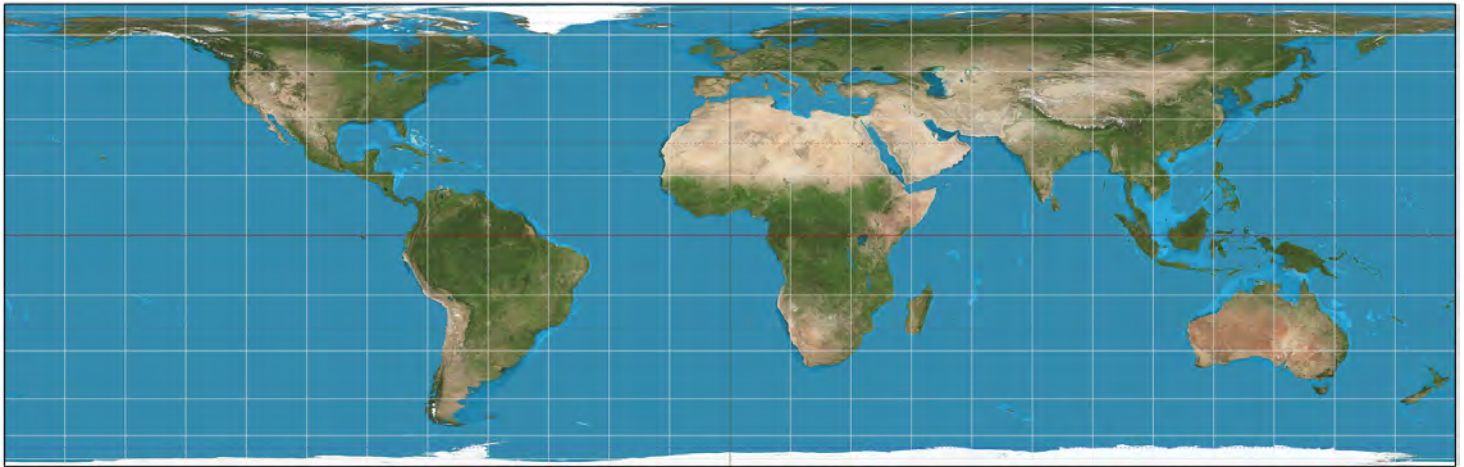
Aitoff Map



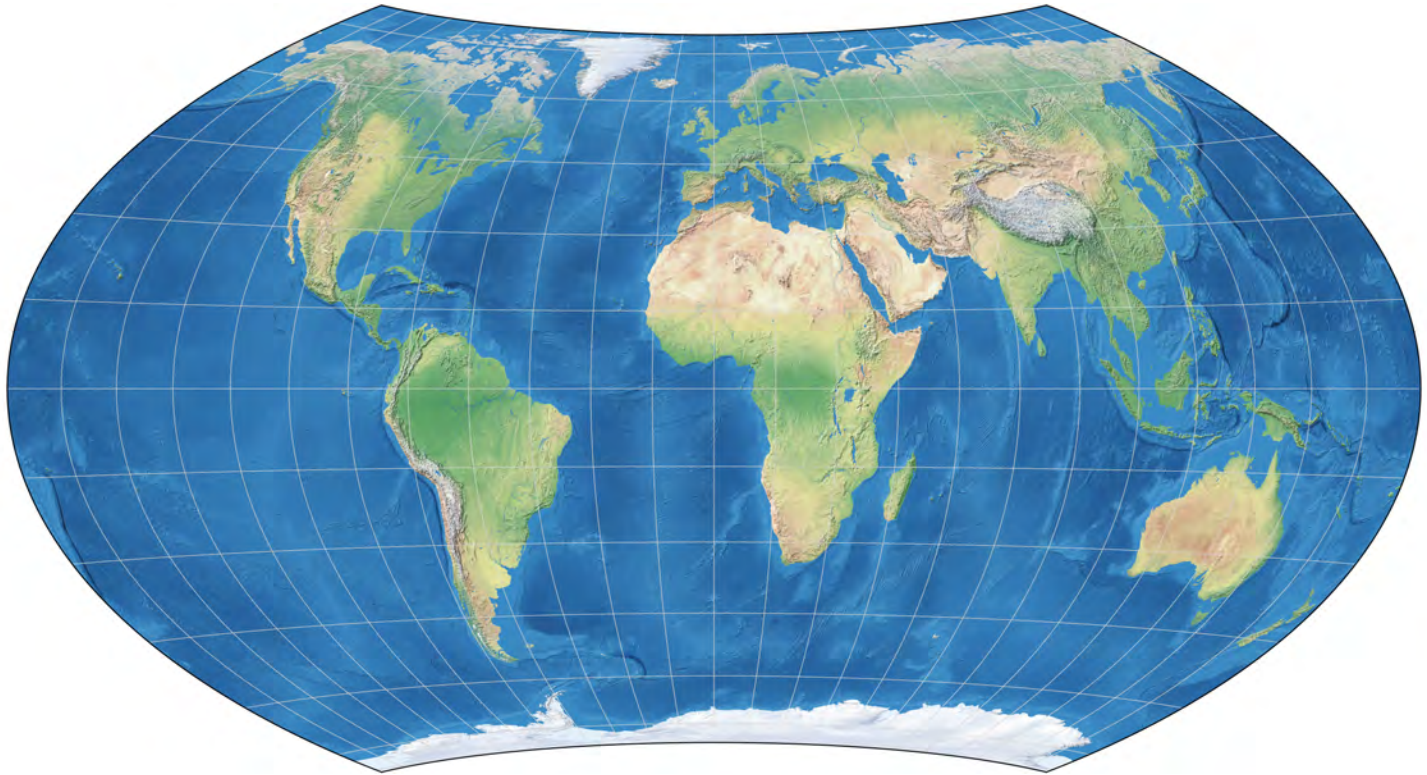
Mollweide Map



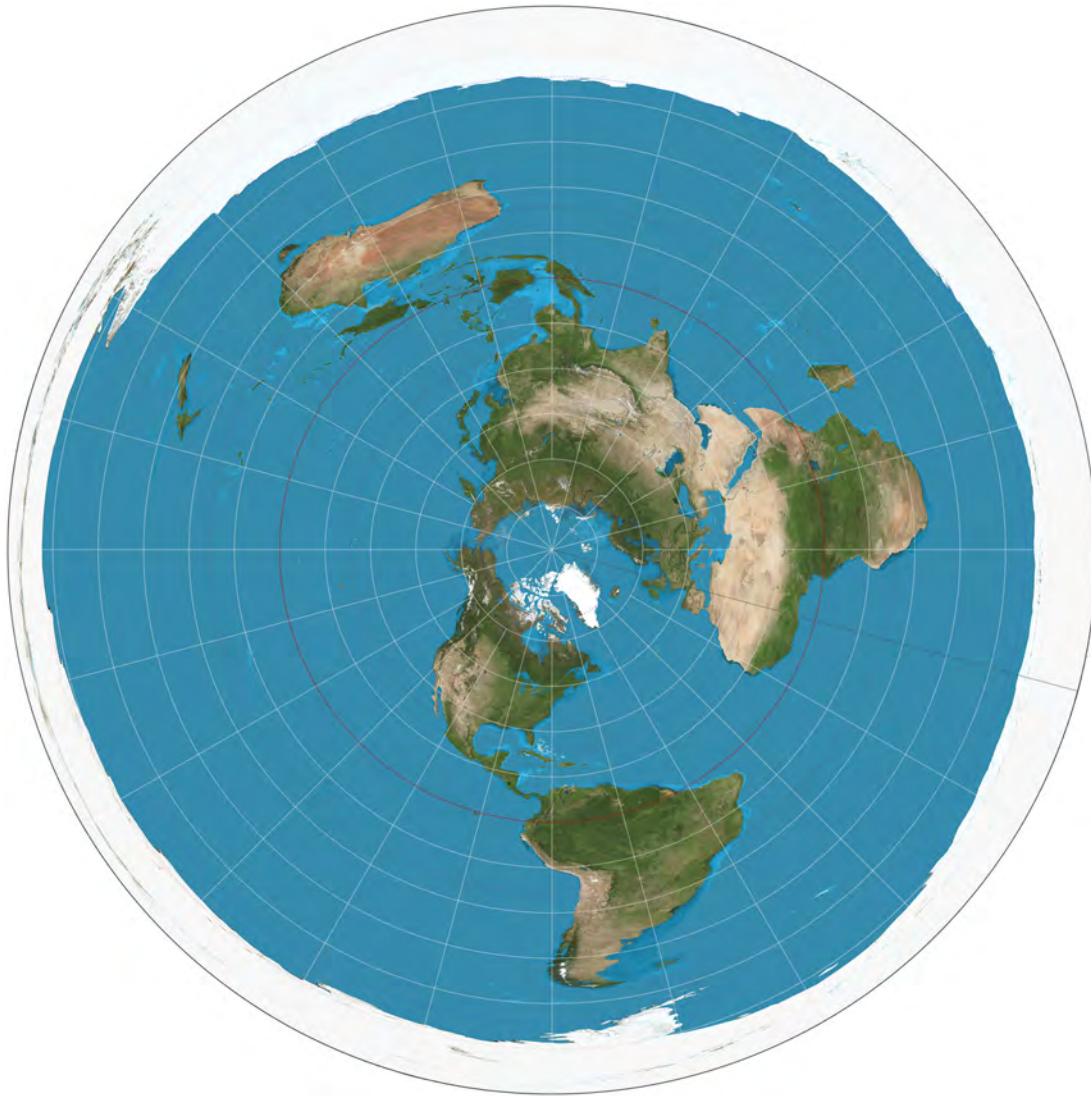
Lambert Map



Wagner Map



Azimuthal Equidistant



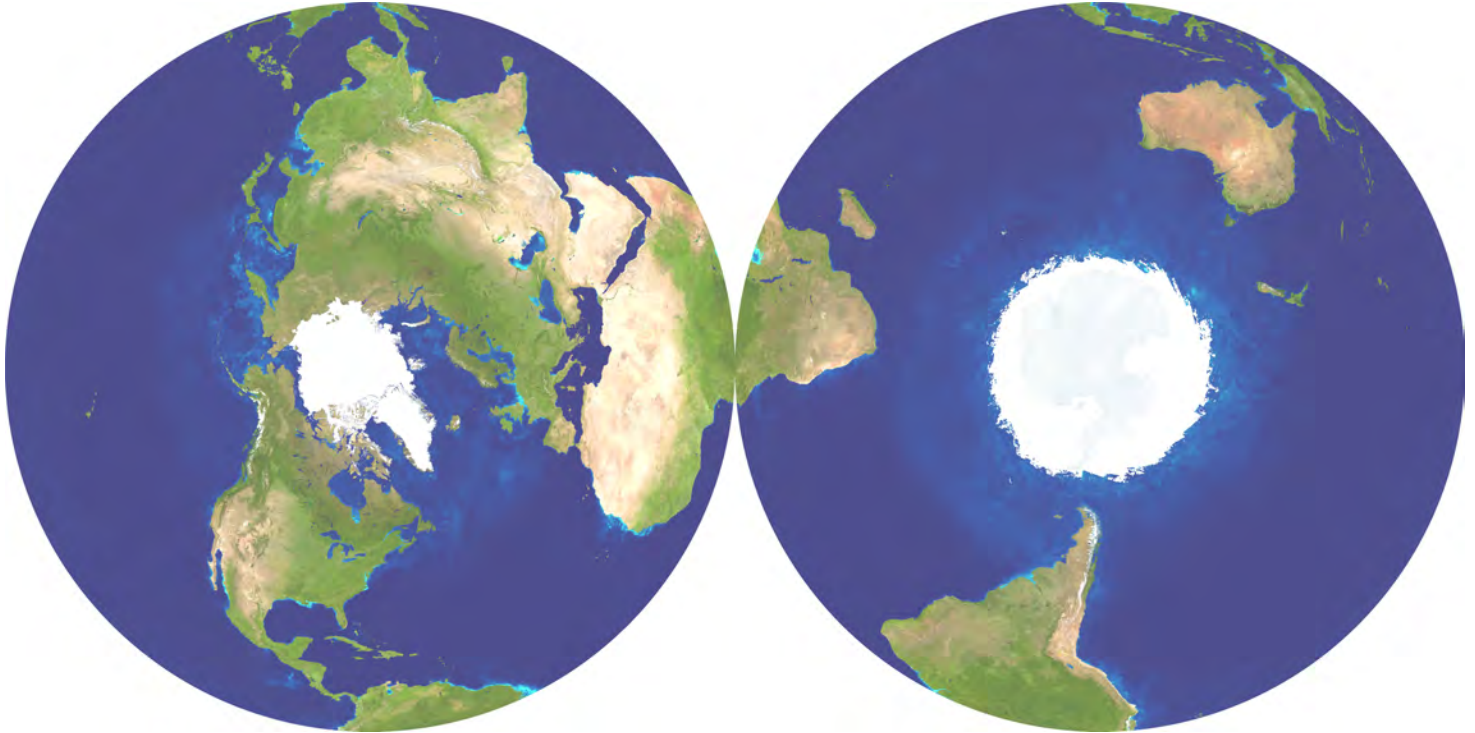
Distortion Metrics

- Skewness
- Flexion
- Isotropy
- Area
- Distances
- Boundary Cuts

Winkel Tripel is “best”

A New Map Projection...

Gott-Goldberg-Vanderbei Map



It's a Two-Sided Map



Let's Minimize Stress

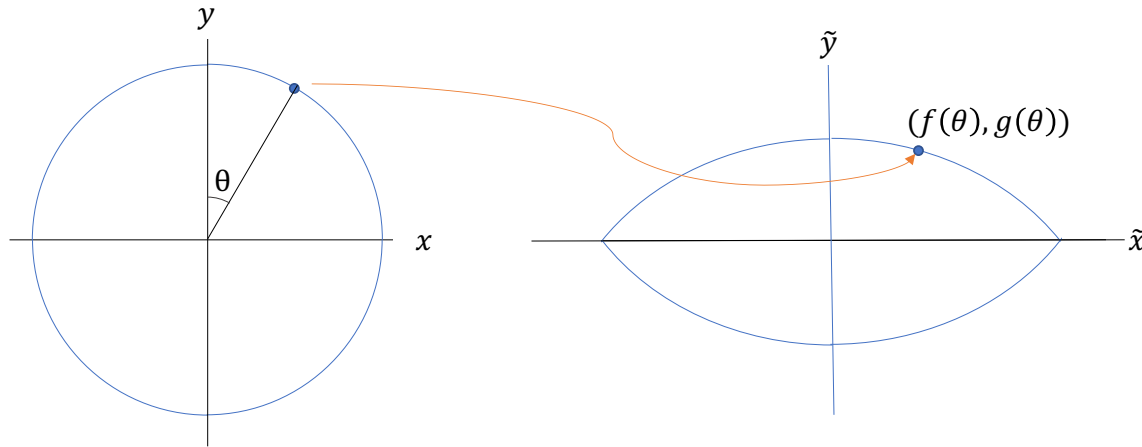
Imagine a Large Rubber Earth Ball



Suppose it has an Expandable Metal Ring Inside



Stretch It



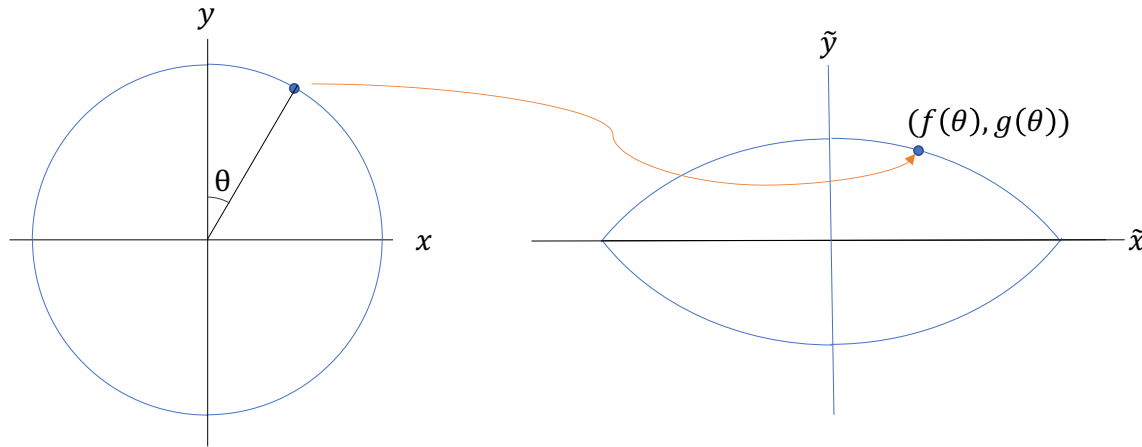
Suppose that the y -axis is the polar axis and hence the equatorial ring is in the (x, z) plane.

Now let's *stretch* the ring so that it has a radius larger than its default value of, say, 1.

Without loss of generality, we can focus on just one longitudinal plane, let's say the one associated with $z = 0$.

As shown above, the geometry of the stretched ball can be described by two functions f and g .

Let's Do Some Math



Let x and y denote the coordinates of the *unstretched* ball and let \tilde{x} and \tilde{y} denote the coordinates of the *stretched* ball.

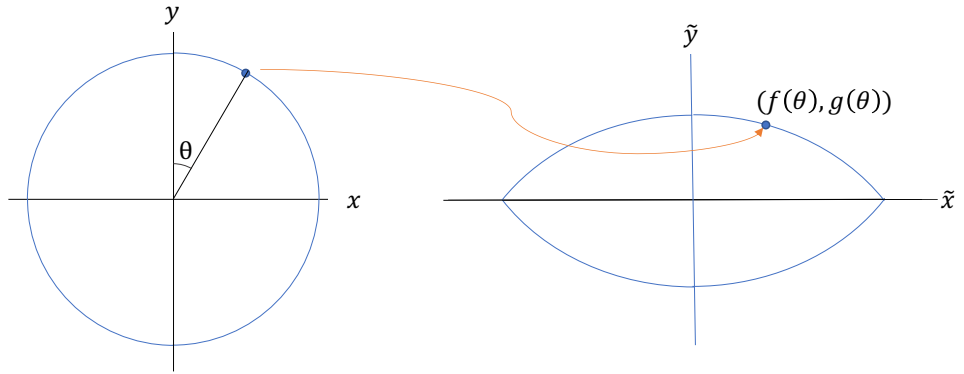
If we let θ denote the angle down from the North Pole, then we have

$$x(\theta) = \sin(\theta), \quad y(\theta) = \cos(\theta) \quad \text{and} \quad \tilde{x}(\theta) = f(\theta), \quad \tilde{y}(\theta) = g(\theta)$$

According to physics, the shape of the stretched ball will be such that the integral over the ball's surface of the magnitude squared of the stress tensor is *minimized*.

Stress

At the point $(\tilde{x}(\theta), \tilde{y}(\theta))$ in the stretched circular slice, let $\sigma(\theta)$ denote the stress in the direction tangent to the circle and let $\rho(\theta)$ denote the stress in the direction perpendicular to the 2-dimensional plane of the slice.



Working with infinitesimal perturbations, we have

$$\|(dx, dy)\| = \sqrt{dx^2 + dy^2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} d\theta = d\theta.$$

and

$$\|(d\tilde{x}, d\tilde{y})\| = \sqrt{d\tilde{x}^2 + d\tilde{y}^2} = \sqrt{f'(\theta)^2 + g'(\theta)^2} d\theta.$$

and from these it is easy to compute $\sigma(\theta)$:

$$\sigma(\theta) = \frac{\|(d\tilde{x}, d\tilde{y})\|}{\|(dx, dy)\|} - 1 = \sqrt{f'(\theta)^2 + g'(\theta)^2} - 1.$$

Computing $\rho(\theta)$ is even easier:

$$\rho(\theta) = \frac{f(\theta)}{\sin(\theta)} - 1.$$

Minimum Stress Problem

$$\min_{f, g} \int_0^{\pi/2} \left(\sigma(\theta)^2 + \rho(\theta)^2 \right) 2\pi \sin(\theta) d\theta$$

where

$$\sigma(\theta) = \sqrt{f'(\theta)^2 + g'(\theta)^2} - 1$$

$$\rho(\theta) = \frac{f(\theta)}{\sin(\theta)} - 1$$

$$g(\theta) = 0, \quad 0 \leq \theta \leq \pi/2$$

$$f(0) = 0$$

$$f'(\theta) \geq 0, \quad 0 \leq \theta \leq \pi/2.$$

Question:

Is the optimal function linear:

$$f(\theta) = c \theta ?$$

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$$f(\theta) = c \theta ?$$

Conjecture:

Maybe

Calculus of Variations

Objective Function:

$$S(f) = \int_0^{\pi/2} \left((f'(\theta) - 1)^2 + \left(\frac{f(\theta)}{\sin(\theta)} - 1 \right)^2 \right) 2\pi \sin(\theta) d\theta$$

Perturbation:

$$\partial f(\theta), \quad 0 \leq \theta \leq \pi/2$$

Critical Points:

$$\lim_{\varepsilon \rightarrow 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 0$$

Compute the Ratio:

$$\begin{aligned}
 & \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} \\
 &= 2\pi \int_0^{\pi/2} \left(\begin{array}{lll} f'(\theta)^2 + 2f'(\theta)\varepsilon \partial f'(\theta) + \varepsilon^2 \partial f'(\theta)^2 & -2f'(\theta) & -2\varepsilon \partial f'(\theta) + 1 \\ -f'(\theta)^2 & +2f'(\theta) & -1 \end{array} \right. \\
 & \quad \left. + \frac{f(\theta)^2}{\sin^2(\theta)} + 2\frac{f(\theta)\varepsilon \partial f(\theta)}{\sin^2(\theta)} + \frac{\varepsilon^2 \partial f(\theta)^2}{\sin^2(\theta)} - 2\frac{f(\theta)}{\sin(\theta)} - 2\frac{\varepsilon \partial f(\theta)}{\sin(\theta)} + 1 \right. \\
 & \quad \left. - \frac{f(\theta)^2}{\sin^2(\theta)} + 2\frac{f(\theta)}{\sin(\theta)} - 1 \right) \sin(\theta) \frac{1}{\varepsilon} d\theta. \\
 &= 2\pi \int_0^{\pi/2} \left(\begin{array}{lll} 2f'(\theta) \partial f'(\theta) + \varepsilon \partial f'(\theta)^2 & -2 \partial f'(\theta) \\ +2\frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} + \frac{\varepsilon \partial f(\theta)^2}{\sin^2(\theta)} & -2\frac{\partial f(\theta)}{\sin(\theta)} \end{array} \right) \sin(\theta) d\theta.
 \end{aligned}$$

Take the Limit:

$$\frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 2\pi \int_0^{\pi/2} \left(\begin{array}{l} 2f'(\theta) \partial f'(\theta) + \varepsilon \partial f'(\theta)^2 - 2 \partial f'(\theta) \\ + 2 \frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} + \varepsilon \frac{\partial f(\theta)^2}{\sin^2(\theta)} - 2 \frac{\partial f(\theta)}{\sin(\theta)} \end{array} \right) \sin(\theta) d\theta$$

$$\lim_{\varepsilon \rightarrow 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 4\pi \int_0^{\pi/2} \left(\begin{array}{l} f'(\theta) \partial f'(\theta) \qquad \qquad \qquad - \partial f'(\theta) \\ + \frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} \qquad \qquad \qquad - \frac{\partial f(\theta)}{\sin(\theta)} \end{array} \right) \sin(\theta) d\theta.$$

Simpler notation...

$$\frac{\partial S}{\partial f} = \lim_{\varepsilon \rightarrow 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon}$$

Critical Point:

Set the differential to zero...

$$\begin{aligned}\frac{\partial \mathcal{S}}{\partial f} &= 4\pi \int_0^{\pi/2} \left(\begin{array}{l} f'(\theta) \partial f'(\theta) \qquad - \partial f'(\theta) \\ + \frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} \qquad - \frac{\partial f(\theta)}{\sin(\theta)} \end{array} \right) \sin(\theta) d\theta \\ &= 4\pi \int_0^{\pi/2} \left((f'(\theta) - 1) \partial f'(\theta) + \left(\frac{f(\theta)}{\sin(\theta)} - 1 \right) \partial f(\theta) \right) \sin(\theta) d\theta \\ &= 0.\end{aligned}$$

Integrate by Parts:

$$\int_0^{\pi/2} (f'(\theta) - 1) \sin(\theta) \partial f'(\theta) d\theta = (f'(\pi/2) - 1) \partial f(\pi/2) - \int_0^{\pi/2} (f''(\theta) \sin(\theta) + (f'(\theta) - 1) \cos(\theta)) \partial f(\theta) d\theta.$$

Substituting this into our equation defining critical points, we get

$$0 = (f'(\pi/2) - 1) \partial f(\pi/2) - \int_0^{\pi/2} \left(f''(\theta) \sin(\theta) + (f'(\theta) - 1) \cos(\theta) - \frac{f(\theta)}{\sin(\theta)} + 1 \right) \partial f(\theta) d\theta.$$

This equation must be equal to zero for all valid choices of the perturbation function ∂f . Hence...

Differential Equation:

$$\sin^2(\theta)f''(\theta) + \sin(\theta)\cos(\theta)f'(\theta) - f(\theta) = \sin(\theta)\cos(\theta) - \sin(\theta)$$

$$f(0) = 0$$

$$f'(\pi/2) = 1.$$

Differential Equation:

$$\sin^2(\theta)f''(\theta) + \sin(\theta)\cos(\theta)f'(\theta) - f(\theta) = \sin(\theta)\cos(\theta) - \sin(\theta)$$

$$f(0) = 0$$

$$f'(\pi/2) = 1.$$

Let's try $f(\theta) = \theta \dots$

$$\sin^2(\theta)f''(\theta) + \sin(\theta)\cos(\theta)f'(\theta) - f(\theta) = \sin(\theta)\cos(\theta) - \theta$$

$$\neq \sin(\theta)\cos(\theta) - \sin(\theta)$$

Almost but no cigar. 🤔

Mathematica

```
s = DSolve[ {Sin[x]^2*y''[x]+Sin[x]*Cos[x]*y'[x]-y[x]==Sin[x]*Cos[x]-Sin[x],  
            y[0]==0, y'[Pi/2]==1}, y[x], x] // FullSimplify  
f[x_]=y[x]/.s[[1]]
```

The output produced by *Mathematica* (with x changed to θ) is

$$f(\theta) = \log(2) \tan(\theta/2) - 2 \cot(\theta/2) \log(\cos(\theta/2)).$$

Matlab

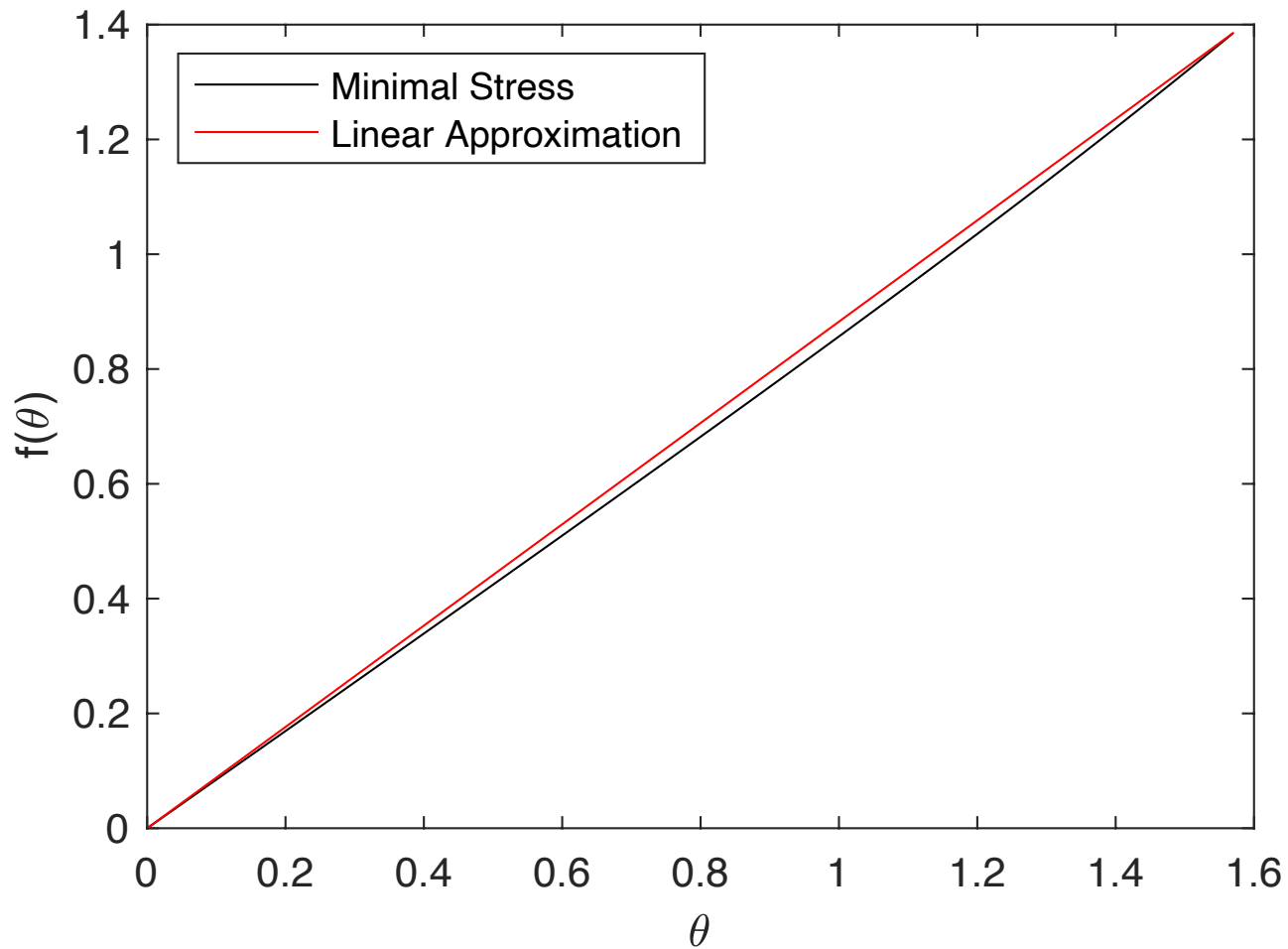
```
syms f(x)  
f1 = diff(f,x);  
f2 = diff(f,x,2);  
ode = sin(x)^2 * f2 + sin(x)*cos(x) * f1 - f == sin(x)*cos(x) - sin(x);  
cond1 = f(0) == 0;  
cond2 = f1(pi/2) == 1;  
conds = [cond1 cond2];  
fSol(x) = dsolve(ode,conds)  
fSim(x) = simplify(fSol(x), 'steps', 14)
```

The output produced by *Matlab* (again with x changed to θ) is

$$f(\theta) = -\frac{\log(\cos(\theta)/4 + 1/4) + 2 \log(e^{i\theta} + 1) \cos(\theta) - \log(2) \cos(\theta) - \theta \cos(\theta)i}{\sin(\theta)}$$

NOTE: These two functions look different, but they are the same.

Almost Linear



Check that it's a Min, not a Max or a Saddle Point

Let's look at the second order differential in every possible perturbational direction...

$$\frac{\partial^2 S}{\partial f^2} = \lim_{\varepsilon \rightarrow 0} \frac{S(f + \varepsilon \partial f) - 2S(f) + S(f - \varepsilon \partial f)}{\varepsilon^2}$$

Let's compute...

$$\begin{aligned} \frac{S(f + \varepsilon \partial f) - 2S(f) + S(f - \varepsilon \partial f)}{\varepsilon^2} &= \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon^2} + \frac{S(f - \varepsilon \partial f) - S(f)}{\varepsilon^2} \\ &= 4\pi \int_0^{\pi/2} \left(\partial f'(\theta)^2 + \frac{\partial f(\theta)^2}{\sin^2(\theta)} \right) \sin(\theta) d\theta \\ &\geq 0. \end{aligned}$$

Ergo, it's a minimum!

THANK YOU

Solving the Differential Equation

Let

$$g(\theta) = \sin(\theta)f(\theta).$$

The differential equation in terms of g :

$$\begin{aligned}\sin(\theta)g''(\theta) - \cos(\theta)g'(\theta) &= \sin(\theta)\cos(\theta) - \sin(\theta) \\ g(0) &= 0 \\ g'(\pi/2) &= 1.\end{aligned}$$

Let

$$h(\theta) = g'(\theta).$$

The differential equation in terms of h :

$$\begin{aligned}\sin(\theta)h'(\theta) - \cos(\theta)h(\theta) &= \sin(\theta)\cos(\theta) - \sin(\theta) \\ h(\pi/2) &= 1.\end{aligned}$$

The solution is:

$$h(\theta) = \sin(\theta) + \log(\cos(\theta) + 1)\sin(\theta).$$

Integrating, we get:

$$g(\theta) = 2\log(2) - (\cos(\theta) + 1)\log(\cos(\theta) + 1)$$

and from that we get:

$$f(\theta) = \frac{2\log(2) - (\cos(\theta) + 1)\log(\cos(\theta) + 1)}{\sin(\theta)}.$$