Stability of Ring Systems

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Linear Stability of Ring Systems, *Astronomical Journal*, *133*:656-664, 2007

Linear Stability of Ring Systems Around Oblate Central Masses, *Advances in Space Research*, *42*:1370–1377, 2008

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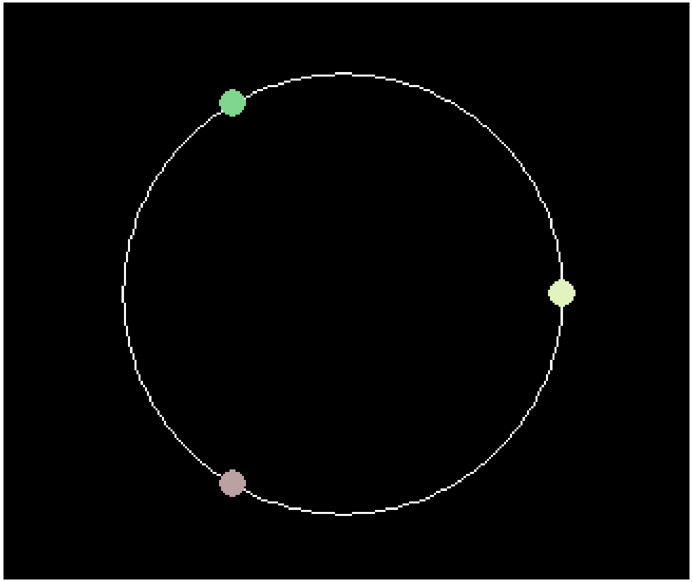
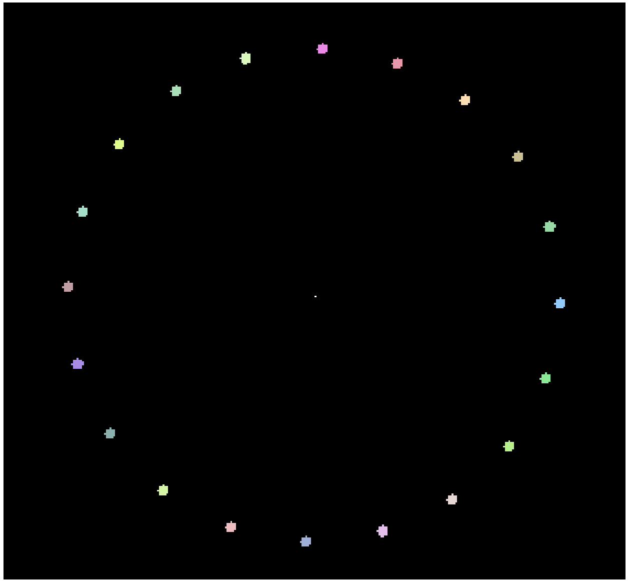
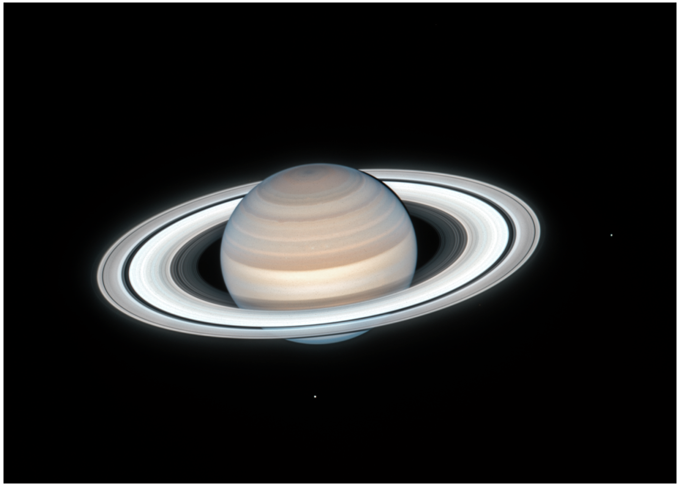
Isolated Ring Systems Are Unstable



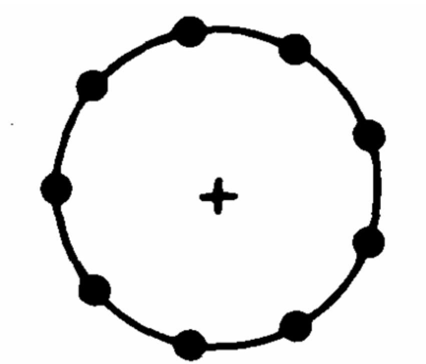


Theorem 1 *The system is stable if and only if* $n = 2$.

Saturn's Rings



Beautiful Saturn



Simplified model of a ring system

In 1859, J.C. Maxwell won the prestigious Adams Prize.

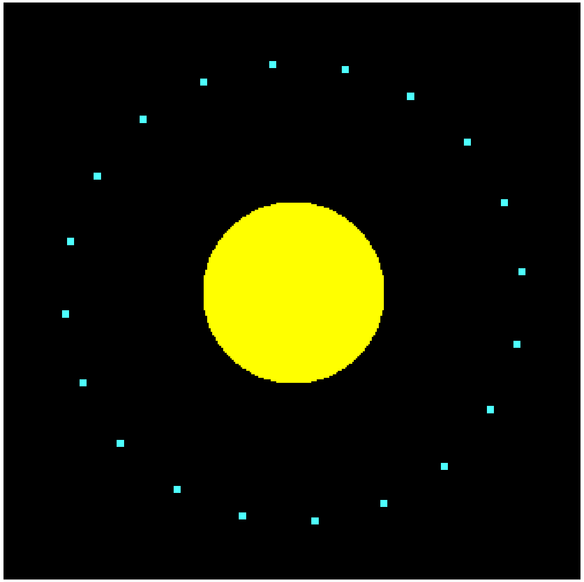
His Results:

- Rings of Saturn must be composed of small particles.
- Modeled the ring as n co-orbital particles of mass m.
- For large n, ring system is stable if

$$\frac{m}{M} \leq \frac{2.298}{n^3}$$

Image From Earth

A Large Central Mass Stabilizes



Saturn and 20 Janus-mass moons

Stable! WHY?

Common misconception: the massive body dominates the dynamics dwarfing the moon-moon interactions.

This is WRONG.

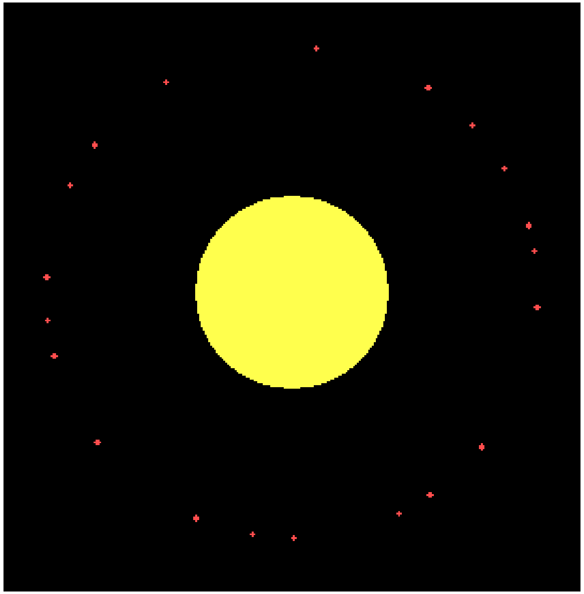
Slight Perturbation

Here, again, are 20 Janus masses

Orbits are initialized to be circular

Distances from Saturn are randomized (only slightly)

Note the effective repulsion!



Main Result

R. J. Vanderbei and K. Kolemen *Linear Stability of Ring Systems*. *Astronomical Journal*, 133:656-664, 2007.

Theorem 2

- *For $2 \leq n \leq 6$, the ring system is unstable.*
- *For $n \geq 7$, the ring system is (linearly) stable if and only if*

$$\frac{m}{M} \leq \frac{\gamma_n}{n^3}.$$

- $\lim_{n\to\infty} \gamma_n = 2.2987$.

Simulation confirms the stability analysis:

n	γ_n	Simulator
2	*	[0.0, 0.007]
6	*	[0.0, 0.025]
7	2.452	[2.45, 2.46]
8	2.4121	[2.41, 2.42]
10	2.3753	[2.37, 2.38]
12	2.3543	[2.35, 2.36]
14	2.3411	[2.34, 2.35]
20	2.3213	[2.32, 2.33]
36	2.3066	[2.30, 2.31]
50	2.3031	[2.30, 2.31]
100	2.2999	[2.30, 2.31]
500	2.2987	

The Formula For γ_n Is Explicit But Ugly

$$
\begin{aligned}
n^3/\gamma_n \;=\; & 2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 5I_n \\
& + \sqrt{\left(2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 4I_n\right)^2 - \frac{9}{4}\left(J_n - \tilde{J}_{n/2,n}\right)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
I_n &= \frac{1}{4}\sum_{k=1}^{n-1} \frac{1}{\sin\left(\pi k/n\right)} \\
J_n &= \frac{1}{4}\sum_{k=1}^{n-1} \frac{1}{\sin^3\left(\pi k/n\right)} \\
\tilde{J}_{j,n} &= \frac{1}{4}\sum_{k=1}^{n-1} \frac{\cos(2\pi k j/n)}{\sin^3(\pi k/n)}
\end{aligned}
$$

Asymptotics

For n large,

$$I_n \;\approx\; \frac{n}{2\pi} \sum_{k=1}^{(n-1)/2} \frac{1}{k} \;\approx\; \frac{n}{2\pi}\; \log(n/2)$$

$$J_n \;\approx\; \frac{n^3}{2\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \;=\; \frac{n^3}{2\pi^3}\,\zeta(3) = 0.01938\; n^3$$

$$\tilde{J}_{n/2,n} \;\approx\; -\frac{3}{4} J_n.$$

Hence,

$$\gamma_n \approx \frac{1}{\frac{7}{8}(13+\sqrt{160})J_n/n^3} \approx 2.2987.$$

Oblateness

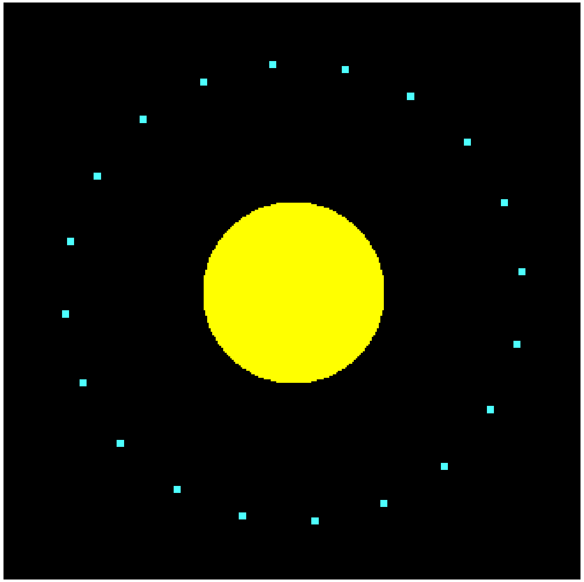
If the central body is oblate with oblateness parameter $\mathcal{J}_2$ and equatorial radius R, a similar analysis yields, for large n,

$$\gamma_n \approx \frac{8}{7} \frac{(1 - \frac{3}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2}{13 - \frac{57}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2 + \sqrt{(13 - \frac{57}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2 - 9(1 - \frac{3}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2}} \frac{n^3}{J_n}$$

For Saturn, $\mathcal{J}_2 = 1.6297 \times 10^{-2}$ and $R/r = 0.3967$. With these values, we get

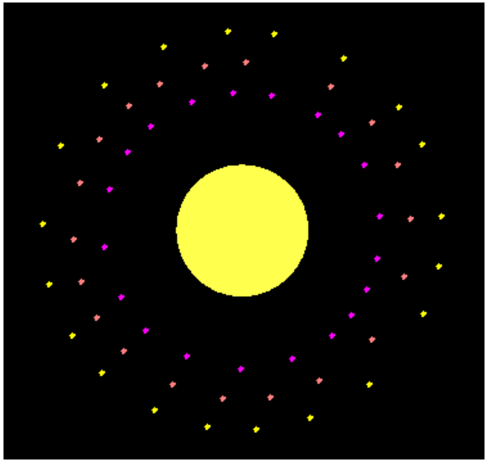
$$\gamma_n \approx 2.2945.$$

From simulator with $n = 60$, 2.280 is stable whereas 2.281 is not.



Rings at Multiple Radii

General principle: it is easier for a body to destabilize bodies at the same radius from the central mass.



Hence, if each of many single rings are stable, then one might expect the entire system to be stable.

Mathematical verification is profoundly difficult—no longer does a single counter-rotation freeze all bodies.

Density Estimate

Let

$$\lambda \;=\; \text{linear density of the masses} = \frac{\text{diam of a boulder}}{\text{separation between boulders}}$$

If δ denotes the boulders' density, then the mass of a boulder is

$$m = (4\pi/3)(\lambda\pi r/n)^3\delta.$$

The density of the boulders in Saturn's rings is about $1/8$ of Earth's density

$$\delta = \frac{1}{8}\frac{M_E}{(4\pi/3)r_E^3}.$$

Recall our stability threshold

$$m \leq 2.298M/n^3.$$

Combining, we get an inequality *without n*:

$$\left(\lambda\pi\frac{r}{r_E}\right)^3 \leq (8)(2.298)\left(\frac{M_S}{M_E}\right)$$

Substituting $r = 120,000$km and $M_S = 95.5M_E$ and solving for λ, we get

$$\lambda \leq 20.4\%.$$

Remark: Gravity scales correctly—a marble orbits a bowling ball every 90 minutes.

References

[1] J.C. Maxwell. *On the Stability of Motions of Saturn's Rings*. Macmillan and Company, Cambridge, 1859.

[2] F. Tisserand. *Traité de Méchanique Céleste*. Gauthier-Villars, Paris., 1889.

[3] C. G. Pendse. The Theory of Saturn's Rings. *Royal Society of London Philosophical Transactions Series A*, 234:145–176, March 1935.

[4] P. Goldreich and S. Tremaine. The dynamics of planetary rings. *Annual Review of Astronomy and Astrophysics*, 20:249–283, 1982.

[5] E. Willerding. Theory of density waves in narrow planetary rings. *AAP*, 161:403–407, June 1986.

[6] H. Salo and C.F. Yoder. The dynamics of coorbital satellite systems. *Astronomy and Astrophysics*, 205:309–327, 1988.

[7] D. J. Scheeres and N. X. Vinh. Linear stability of a self-gravitating ring. *Celestial Mechanics and Dynamical Astronomy*, 51:83–103, 1991.

[8] P. Hut, J. Makino, and S. McMillan. Building a better leapfrog. *The Astrophysical Journal—Letters*, 443:93–96, 1995.

[9] P. Saha and S. Tremaine. Long-term planetary integration with individual time steps. *Astronomical Journal*, 108:1962, 1994.

[10] H. Salo. Simulations of dense planetary rings. iii. self-gravitating identical particles. *Icarus*, 117:287–312, 1995.

Appendix: Some Details

Complex Notation is Simple

Equation of motion for $j = 0, \ldots, n-1$

$$\ddot{z}_j = GM \frac{z_n - z_j}{|z_n - z_j|^3} + \sum_{k \neq j,n} Gm \frac{z_k - z_j}{|z_k - z_j|^3}.$$

About center of mass

$$z_n = -\frac{m}{M} \sum_{j=0}^{n-1} z_j.$$

Equilibrium point

$$\begin{aligned} z_j(t) &= re^{i(\omega t + 2\pi j/n)}, \qquad j = 0, \ldots, n-1 \\ z_n(t) &= 0, \end{aligned}$$

where

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{4r^3} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)}.$$

Linear Stability Analysis

Counter rotate (and map to positive real axis):

$$w_j = e^{-i(\omega t + 2\pi j/n)} z_j.$$

Treating w_j and $\bar{w}_j$ as independent variables, put

$$W_j = \begin{bmatrix} w_j \\ \bar{w}_j \end{bmatrix}.$$

Linearize equation of motion around equilibrium point:

$$\frac{d}{dt}\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} \approx \left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega \end{array}\right] \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}$$

Stability is Determined by Eigenvalues of $4n \times 4n$ System

$$\left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega \end{array}\right] \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}.$$

First $2n$ equations give

$$\delta \dot{W}_j = \lambda \delta W_j$$

Substituting, we get a *block circulant matrix*:

$$\begin{bmatrix} D & N_1 & \cdots & N_{n-1} \\ N_{n-1} & D & \cdots & N_{n-2} \\ \vdots & \vdots & & \vdots \\ N_1 & N_2 & \cdots & D \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} + \lambda \begin{bmatrix} \Omega & & & \\ & \Omega & & \\ & & \ddots & \\ & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \lambda^2 \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}.$$

Block Circulant Matrix

Look for solutions of the form:

$$\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \begin{bmatrix} \xi \\ \rho_j \xi \\ \vdots \\ \rho_j^{n-1} \xi \end{bmatrix},$$

where ρ_j is an n-th root of unity

$$\rho_j = e^{2\pi i j/n}.$$

The $2n \times 2n$ system then reduces to n 2×2 systems the determinant of which must vanish:

$$\det\left(D + \sum_{k=1}^{n-1} \rho_j^k N_k + \lambda \Omega - \lambda^2 I\right) = 0.$$

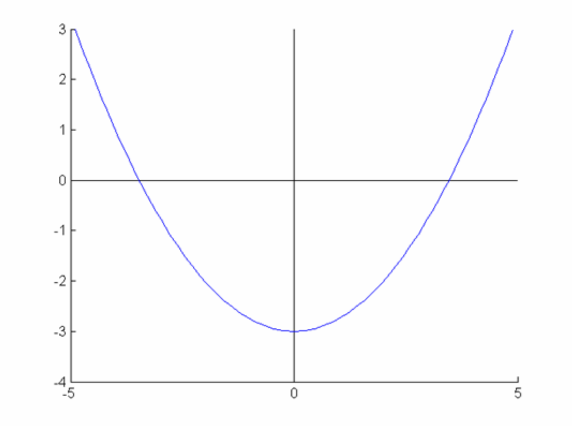
Replacing λ with $i\lambda$, we get a characteristic polynomial with real coefficients

$$f(\lambda) = \lambda^4 + A_j \lambda^2 + B_j \lambda + C_j = 0.$$

Find when this equation has 4 real roots.

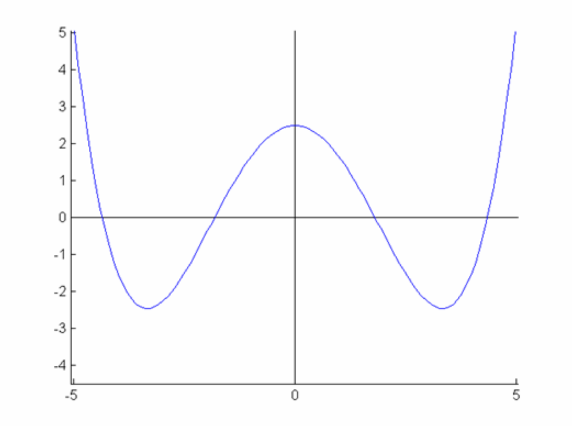
Counting Real Roots of $f(\lambda) = \lambda^4 + A_j\lambda^2 + B_j\lambda + C_j = 0$

For $2 \le n \le 6$ and $j = 1$, $f(\lambda)$ has this form:



Hence, there can be at most 2 real roots and so the system is always unstable.

For $n \ge 7$ and all j, $f(\lambda)$ has this form:



Hence, there can be 4 real roots and so we have the *possibility of stability*.

If $j = n/2$ has four real roots, then so do all other polynomials.

Details are tedious, but analysis of the $j = n/2$ case produces the threshold γ_n given earlier.