Non-Redundant Aperture Masking

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2009 November 6

TPF Group Meeting

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Forget PICTURES

Today we're doing MATH

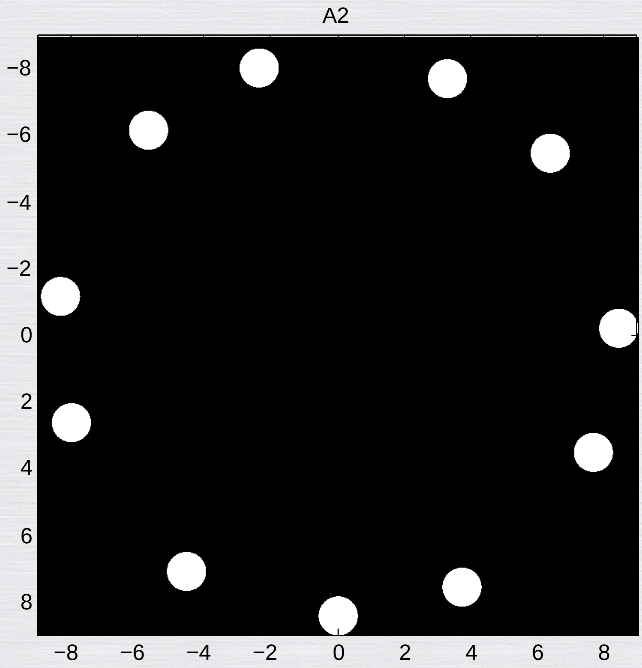
Aperture Mask

Consider a telescope with focal length f and an aperture that has been masked into a disjoint collection of small circular subapertures:

$$A(x, y) = \sum_k A_k(x, y),$$

where

$$A_k(x, y) = 1_{(x-x_k)^2+(y-y_k)^2 \leq a^2/4}$$



is the *indicator function* of a circular subaperture centered at (x_k, y_k) and having diameter a.

Electric Field at Image Plane

The monochromatic electric field at the image plane is then given by

$$
\begin{aligned}
E(\xi,\eta) &= \iint e^{2\pi i(x\xi+y\eta)/\lambda f} A(x,y)\,dx\,dy = \sum_k \iint e^{2\pi i(x\xi+y\eta)/\lambda f} A_k(x,y)\,dx\,dy \\
&= \sum_k \iint_{\bigcirc} e^{2\pi i((x_k+x)\xi+(y_k+y)\eta)/\lambda f}\,dx\,dy = \sum_k e^{2\pi i(x_k\xi+y_k\eta)/\lambda f} \iint_{\bigcirc} e^{2\pi i(x\xi+y\eta)/\lambda f}\,dx\,dy \\
&= E_{a,f}(\xi,\eta) \sum_k e^{2\pi i(x_k\xi+y_k\eta)/\lambda f},
\end{aligned}
$$

where λ is the wavelength of the light, $\iint_{\bigcirc}$ denotes integration over the set $\{(x,y) : x^2+y^2 \le a^2/4\}$, and

$$
E_{a,f}(\xi,\eta) = \iint_{\bigcirc} e^{2\pi i(x\xi+y\eta)/\lambda f}\,dx\,dy = \frac{a}{2\rho} J_1(\pi a\rho/\lambda f)
$$

is the electric field associated with the usual Airy pattern for an aperture of diameter a (note that $\rho = \sqrt{\xi^2+\eta^2}$ simply denotes radius in the image plane).

The Image

As usual, the image is the magnitude squared of the electric field

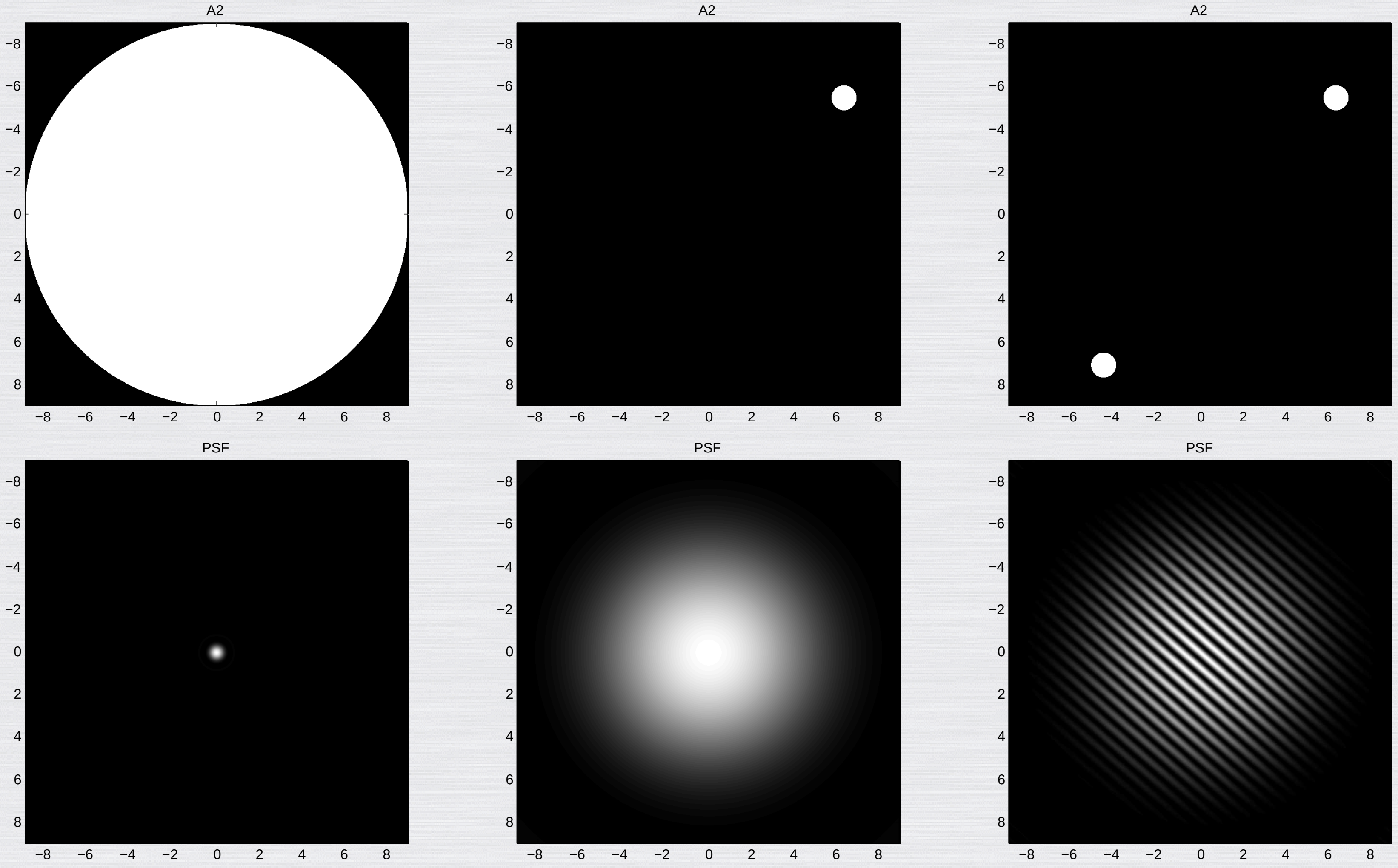
$$
\begin{aligned}
I(\xi,\eta) = |E(\xi,\eta)|^2 &= P_{a,f}(\rho) \sum_{k,k'} e^{2\pi i((x_k - x_{k'})\xi + (y_k - y_{k'})\eta)/\lambda f} \\
&= P_{a,f}(\rho) \left(n + 2 \sum_{k<k'} \cos(2\pi((x_k - x_{k'})\xi + (y_k - y_{k'})\eta)/\lambda f) \right),
\end{aligned}
$$

where

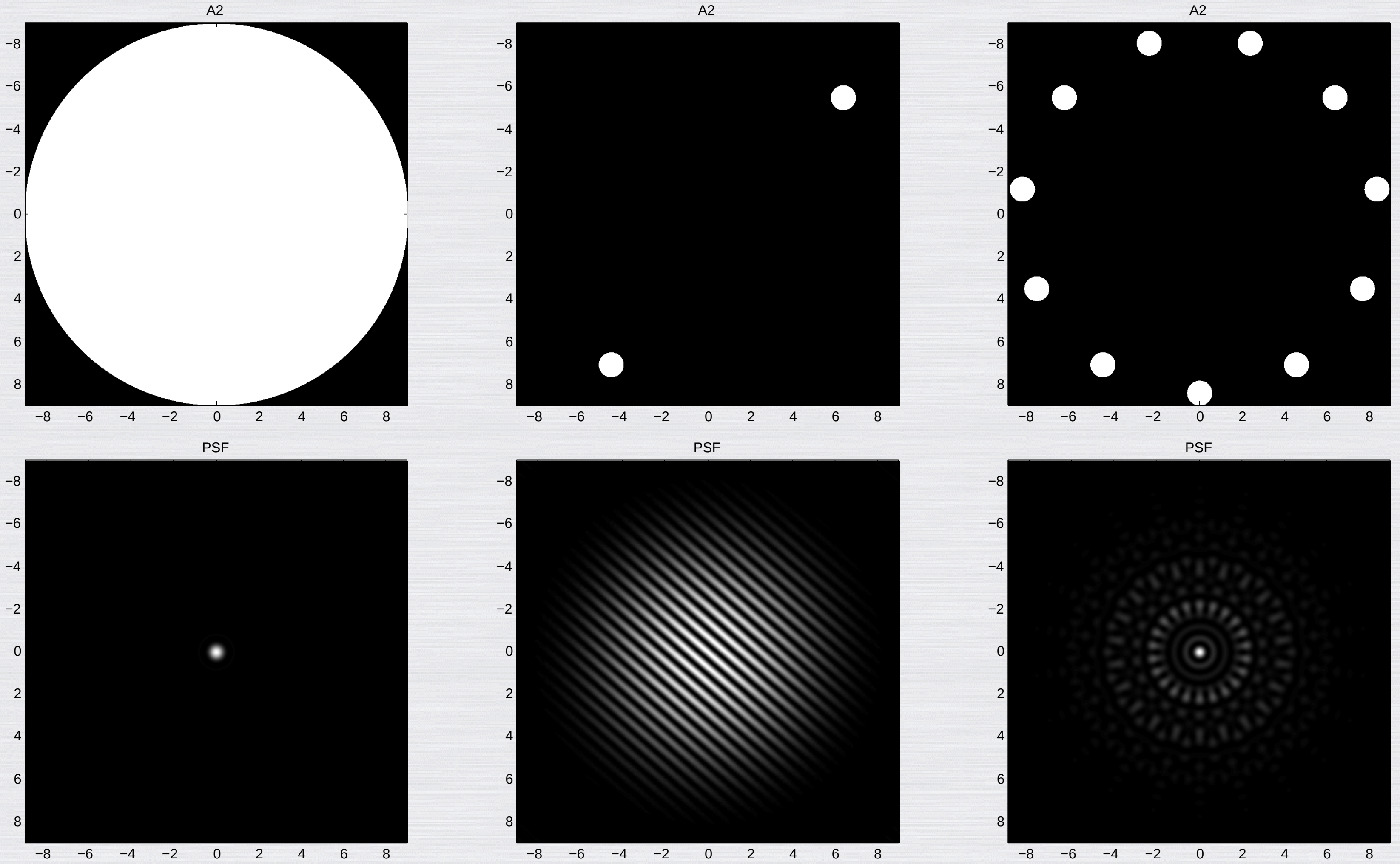
$$
P_{a,f}(\rho) = E^2_{a,f}(\xi,\eta) = \frac{a^2}{4\rho^2} J_1^2(\pi a \rho / \lambda f)
$$

is just the "wide" PSF of associated with the small apertures. Note that the image is this broad PSF modulated by tight fringe patterns associated with the longer baselines ($(x_k - x_{k'}, y_k - y_{k'})$).

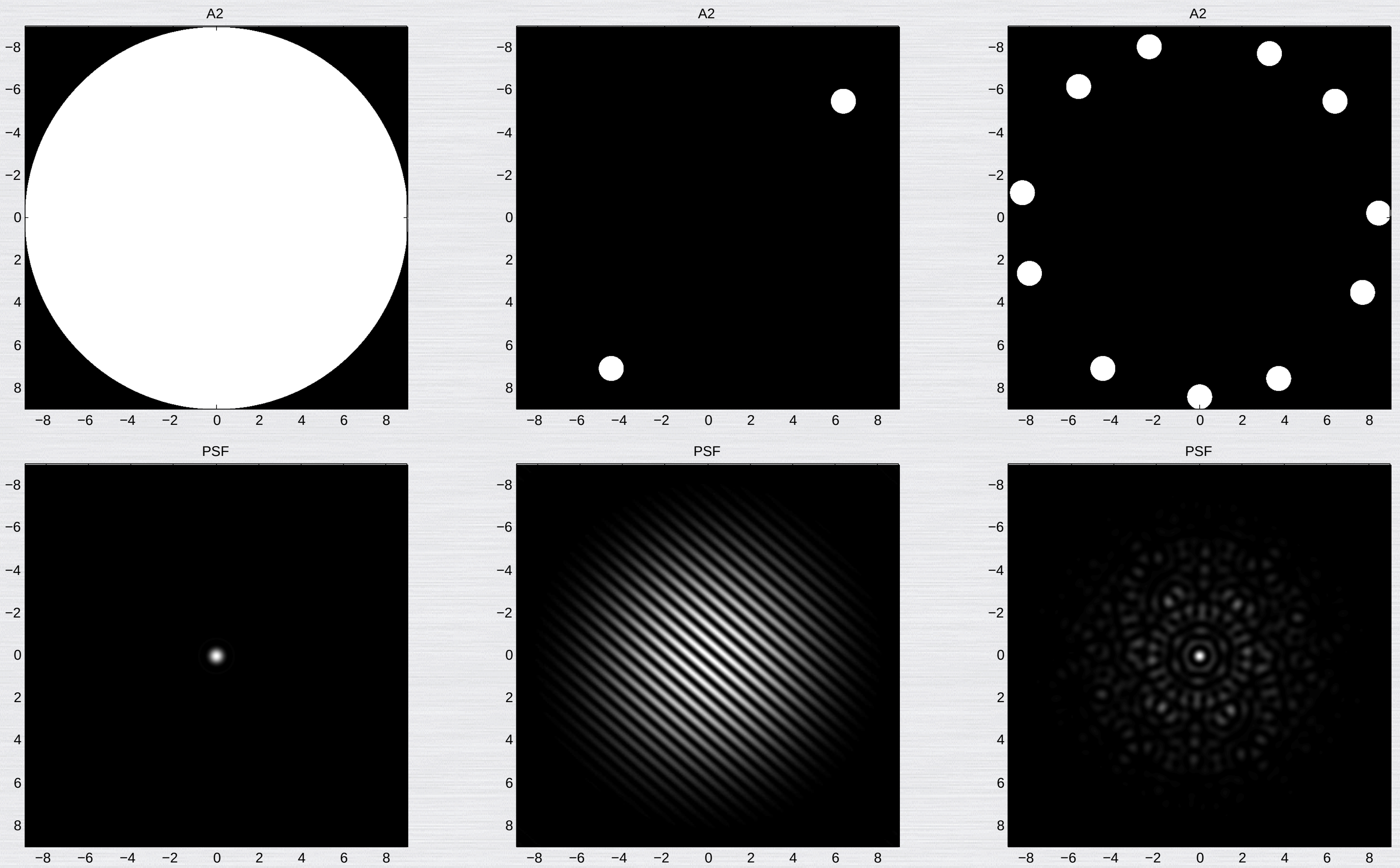
Full Aperture, One Subaperture, Two Subs



Full Aperture, Two Subapertures, Eleven Subs



Full Aperture, Two Subapertures, Eleven Subs



Phase Errors

Now, suppose that atmospheric seeing has introduced some phase errors across the large aperture. For simplicity, we assume that the subapertures are small enough that we can regard the phase errors as fixed over each subaperture. Then, the electric field formula given above has to be modified to account for the phase shifts:

$$\begin{aligned} E(\xi,\eta) &= \iint e^{2\pi i(x\xi+y\eta)/\lambda f} e^{2\pi i\phi(x,y)} A(x,y)dxdy \\ &= \cdots = E_{a,f}(\xi,\eta) \sum_k e^{2\pi i(x_k\xi+y_k\eta)/\lambda f} e^{2\pi i\phi_k/\lambda}, \end{aligned}$$

where $\phi_k = \phi(x_k, y_k) \approx \phi(x, y)$ for all (x, y) in the k-th subaperture. And, the image is the magnitude squared of the electric field:

$$I(\xi,\eta) = P_{a,f}(\rho) \sum_{k,k'} e^{2\pi i((x_k-x_{k'})\xi+(y_k-y_{k'})\eta)/\lambda f + 2\pi i(\phi_k-\phi_{k'})/\lambda}$$

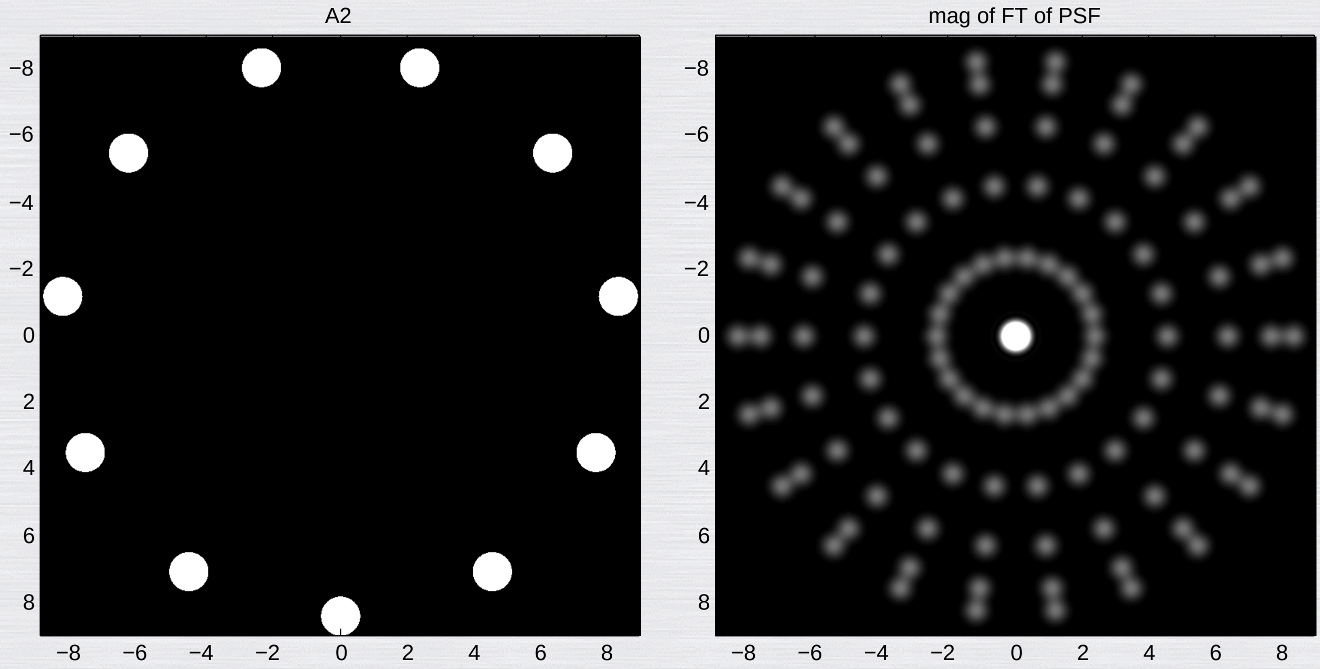
Fourier Analysis

Non-redundant aperture masking makes it possible to automatically remove the phase distortions introduced by the aperture. To do this, one starts by taking the Fourier transform of the image:

$$
\begin{aligned}
F(x,y) &= \iint e^{2\pi i(x\xi+y\eta)/\lambda f} I(\xi,\eta)\,d\xi\,d\eta \\
&= \iint e^{2\pi i(x\xi+y\eta)/\lambda f} P_{a,f}(\rho) \sum_{k,k'} e^{2\pi i((x_k-x_{k'})\xi+(y_k-y_{k'})\eta)/\lambda f+2\pi i(\phi_k-\phi_{k'})/\lambda}\,d\xi\,d\eta \\
&= \sum_{k,k'} e^{2\pi i(\phi_k-\phi_{k'})/\lambda} \iint e^{2\pi i[(x+x_k-x_{k'})\xi+(y+y_k-y_{k'})\eta]/\lambda f} P_{a,f}(\xi,\eta)\,d\xi\,d\eta \\
&= \sum_{k,k'} e^{2\pi i(\phi_k-\phi_{k'})/\lambda} \widehat{P}_{a,f}(x+x_k-x_{k'}, y+y_k-y_{k'}).
\end{aligned}
$$

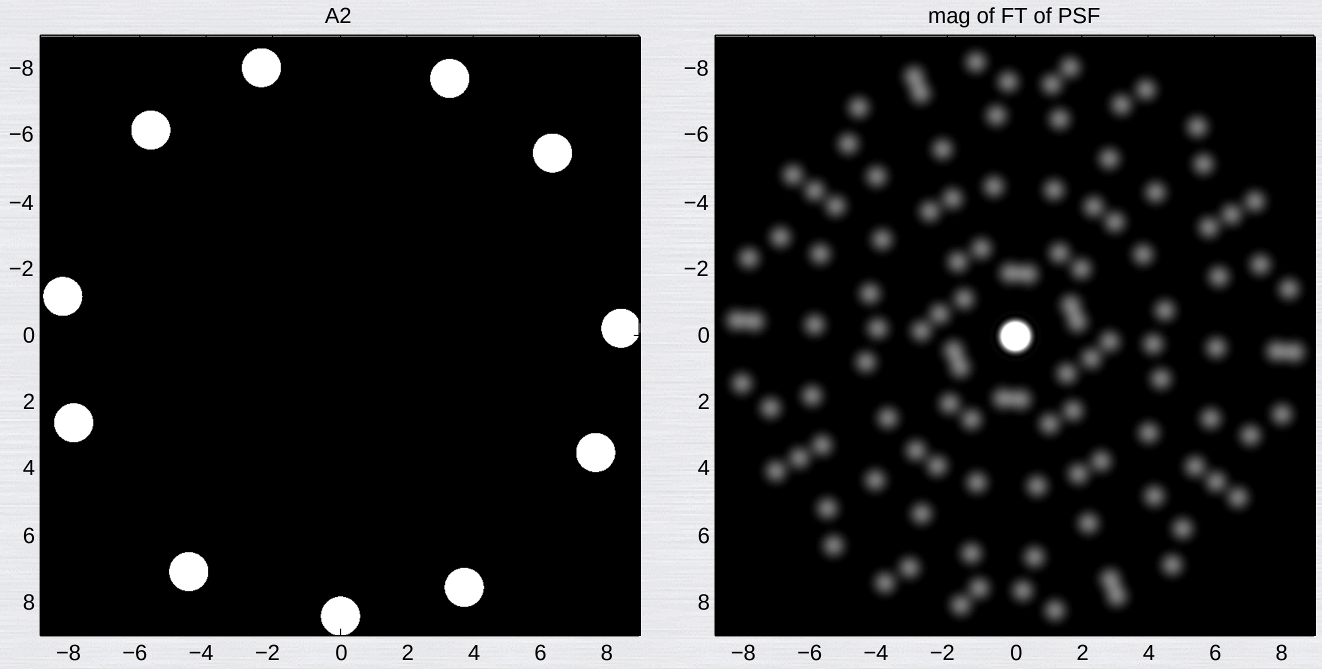
Fourier Transform of Image—Evenly Spaced

No Phase Errors



Fourier Transform of Image—Irregularly Spaced

No Phase Errors



Non-Redundancy and Closure Phase

The function $\widehat{P}_{a,f}$ is the subaperture pupil convolved with itself. Since the pupil has compact support (a disk of diameter a), so does its convolution with itself (having diameter $2a$). Therefore, $F(x,y)$ is a sum of n^2 terms with each term having compact support. The pupil mask is called *non-redundant* if these terms have disjoint support for all $k \neq k'$. In that case, we can extract $n^2 - n$ distinct terms and register them (i.e., center them at $(0,0)$):

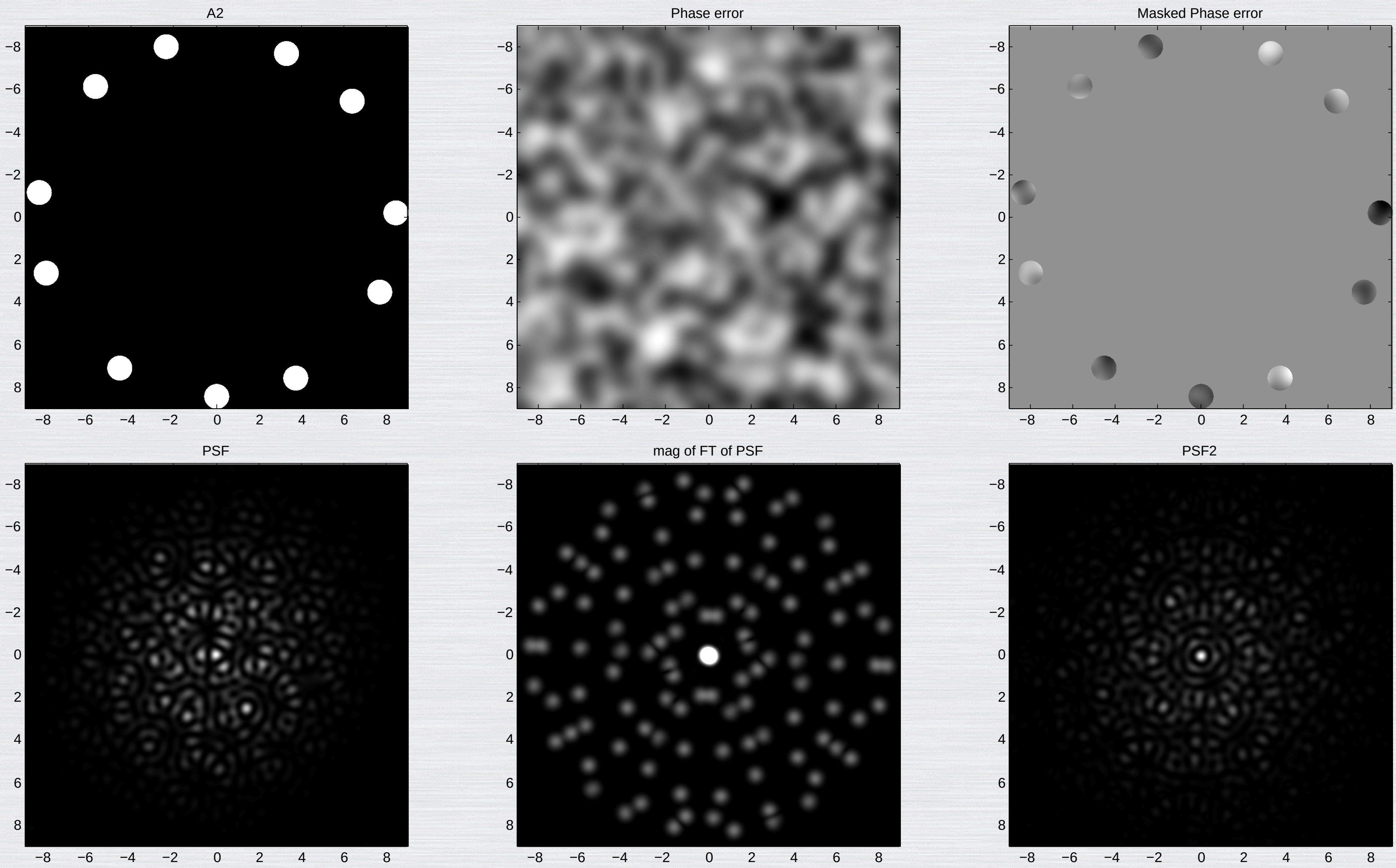
$$F_{k,k'}(x,y) = e^{2\pi i(\phi_k - \phi_{k'})/\lambda} \widehat{P}_{a,f}(x,y), \qquad k,k' = 1,2,\ldots,n, \quad (k \neq k').$$

We can remove the phase errors by multiplying these functions pairwise (I don't understand at the moment why the literature says we have to do a three-factor multiplication):

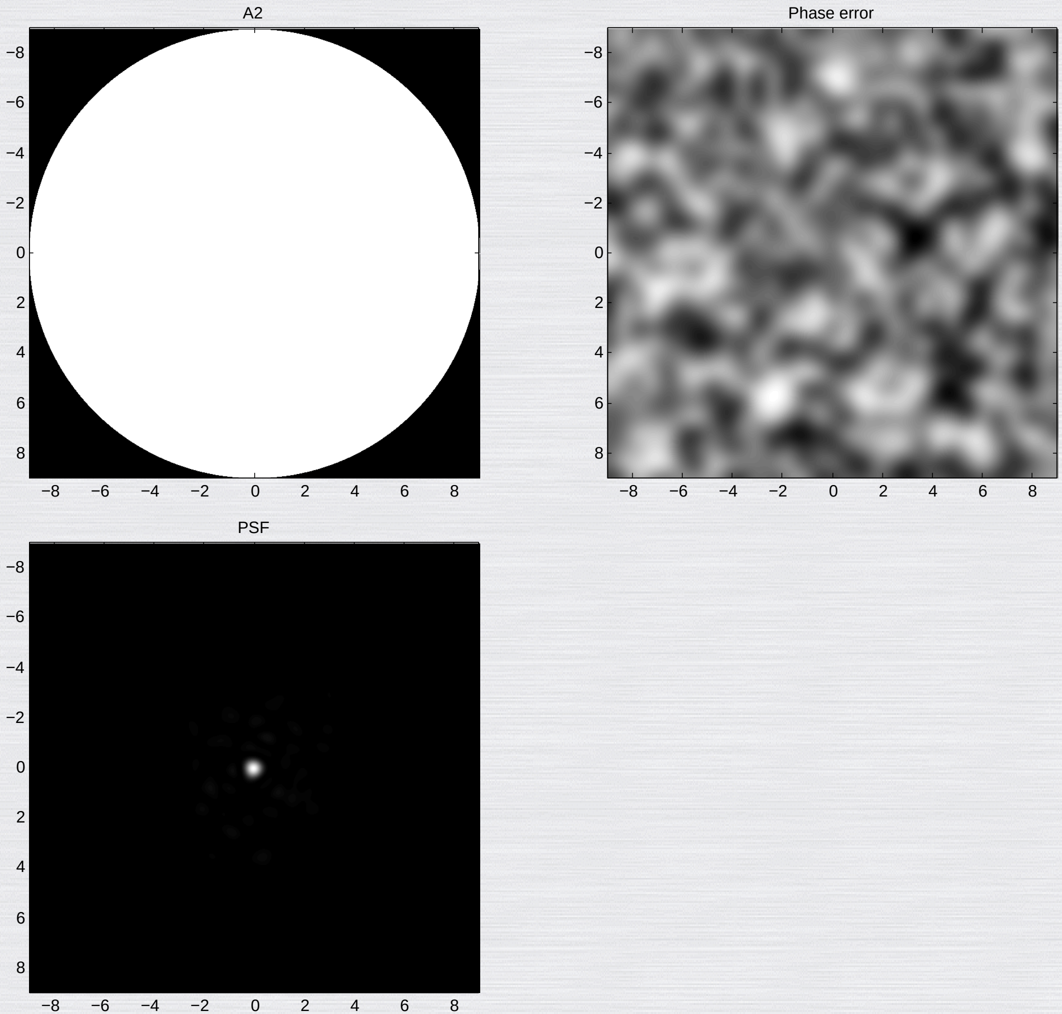
$$F_{k,k'}(x,y) F_{k',k}(x,y) = \widehat{P}^2_{a,f}(x,y).$$

Next, we take the square root and rebuild $F(x,y)$ using these corrected factors and then inverse Fourier transform to get back to a real image.

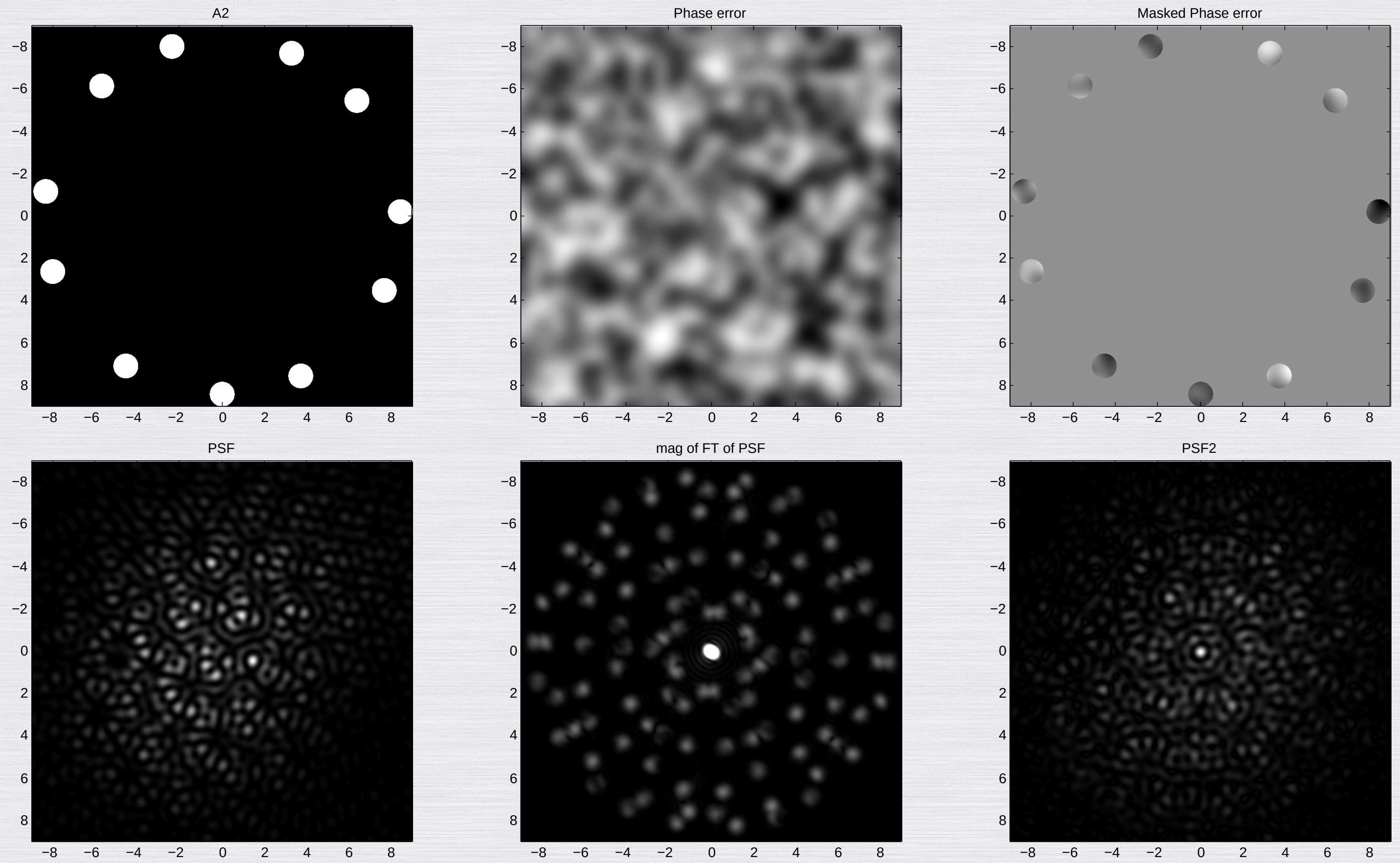
Half-Wave Error Peak-To-Valley



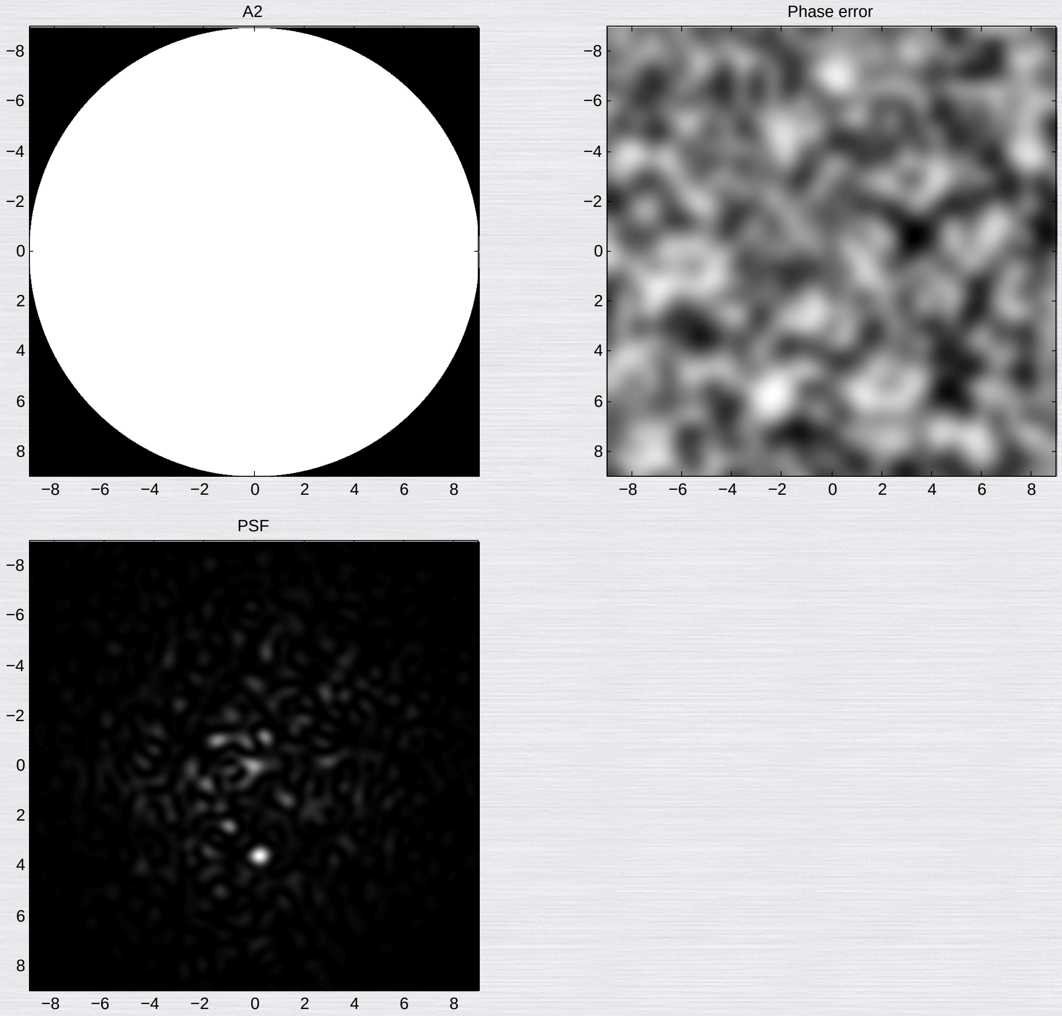
Half-Wave Error Peak-To-Valley, Full Aperture



One-Wave Error Peak-To-Valley



One-Wave Error Peak-To-Valley, Full Aperture



Final Remarks/Questions

- No photon noise
- Monochromatic
- What is the optimal distribution of subapertures?
- Is NRM better than *lucky imaging*?
- Is NRM better than *dynamic deconvolution*?

Off-Axis Source: Angle $= (\alpha, \beta)$ Radians

Image-Plane Electric Field:

$$E(\xi,\eta) = E_{a,f}(\xi + \alpha f, \eta + \beta f) \sum_k e^{2\pi i(x_k\xi + y_k\eta)/\lambda f} e^{2\pi i \phi_k/\lambda} e^{2\pi i(\alpha x_k + \beta y_k)/\lambda}$$

Image:

$$P(\xi,\eta) = P_{a,f}(\xi+\alpha f, \eta+\beta f) \sum_{k,k'} e^{2\pi i\left((x_k - x_{k'})\xi + (y_k - y_{k'})\eta\right)/\lambda f} e^{2\pi i\left(\phi_k - \phi_{k'}\right)/\lambda} e^{2\pi i\left(\alpha(x_k - x_{k'}) + \beta(y_k - y_{k'})\right)/\lambda}$$

Fourier Transform of Image:

$$\begin{aligned} F(x,y) &= \sum_{k,k'} e^{2\pi i(\phi_k - \phi_{k'})/\lambda} e^{2\pi i(\alpha(x_k - x_{k'}) + \beta(y_k - y_{k'}))/\lambda} \widehat{P}_{a,f,\alpha,\beta}(x + x_k - x_{k'}, y + y_k - y_{k'}) \\ &= \sum_{k,k'} F_{k,k'}(x + x_k - x_{k'}, y + y_k - y_{k'}) \end{aligned}$$

where

$$F_{k,k'}(x,y) = e^{2\pi i(\phi_k - \phi_{k'})/\lambda} e^{2\pi i(\alpha(x_k - x_{k'}) + \beta(y_k - y_{k'}))/\lambda} \widehat{P}_{a,f,\alpha,\beta}(x,y)$$

and $\widehat{P}_{a,f,\alpha,\beta}$ is the Fourier transform of $(\xi,\eta) \to P_{a,f}(\xi + \alpha f, \eta + \beta f)$.
Multiplying $F_{k,k'}$ by $F_{k',k}$ wipes out both *phase error* and *tilt*.

Two Sources: On-Axis and Angle = (α, β) Radians

Image:

$$
\begin{aligned}
P(\xi,\eta) \;&=\; c_0 P_{a,f}(\xi,\eta) \sum_{k,k'} e^{2\pi i\left((x_k - x_{k'})\xi + (y_k - y_{k'})\eta\right)/\lambda f} e^{2\pi i\left(\phi_k - \phi_{k'}\right)/\lambda} \\
&\quad + \; c_1 P_{a,f}(\xi + \alpha f, \eta + \beta f) \sum_{k,k'} e^{2\pi i\left((x_k - x_{k'})\xi + (y_k - y_{k'})\eta\right)/\lambda f} e^{2\pi i\left(\phi_k - \phi_{k'}\right)/\lambda} e^{2\pi i\left(\alpha(x_k - x_{k'}) + \beta(y_k - y
\end{aligned}
$$

Fourier Transform of Image:

$$
\begin{aligned}
F(x,y) \;&=\; \sum_{k,k'} e^{2\pi i(\phi_k - \phi_{k'})/\lambda} \Big(c_0 \widehat{P}_{a,f}(x + x_k - x_{k'}, y + y_k - y_{k'}) \\
&\qquad + c_1 e^{2\pi i(\alpha(x_k - x_{k'}) + \beta(y_k - y_{k'}))/\lambda} \widehat{P}_{a,f,\alpha,\beta}(x + x_k - x_{k'}, y + y_k - y_{k'}) \Big) \\
&=\; \sum_{k,k'} F_{k,k'}(x + x_k - x_{k'}, y + y_k - y_{k'})
\end{aligned}
$$

where

$$
F_{k,k'}(x,y) = e^{2\pi i(\phi_k - \phi_{k'})/\lambda} \left(c_0 \widehat{P}_{a,f}(x,y) + c_1 e^{2\pi i(\alpha(x_k - x_{k'}) + \beta(y_k - y_{k'}))/\lambda} \widehat{P}_{a,f,\alpha,\beta}(x,y) \right).
$$

I don't see how any clever products (either two-way, three-way, or higher) can eliminate the phase errors and still allow one to recover "cleaned" versions of the $F_{k,k'}$'s.