Optimization

in

Engineering Design

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Engineering = Optimization

Make something.

Make it better.

Make it better yet.

Make it still better.

⋮

Types of Optimization Problems

Constrained vs. unconstrained.

Convex vs. nonconvex.

Smooth vs. nonsmooth.

Continuous vs. discrete (integer).

Local vs. global.

My main interest: finding *locally optimal* solutions to *nonconvex, constrained, smooth* optimization problems in which all variables are assumed to be *continuous* (based on LOQO).

Types of Algorithms

Black-box-around-legacy-code vs. start-from-scratch (Dennis vs. Betts).

My main interest: the *start-from-scratch* approach (based on AMPL).

LOQO: An algorithm for constrained optimization

LOQO solves problems in the following form:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & b \leq h(x) \leq b + r, \\ & l \leq x \leq u \end{array}$$

The functions $f(x)$ and $h(x)$ must be twice differentiable (at least at points of evaluation).

The standard *interior-point paradigm* is used:

- Add slacks.
- Replace nonnegativities (e.g., $w \geq 0$) with parametrized logarithmic barrier terms ($-\mu \log w$) in objective function.
- Write first-order optimality conditions.
- Rewrite optimality conditions in primal-dual symmetric form.
- Use Newton's method to get search directions...

Interior-Point Paradigm Continued

- Use Newton's method to get search directions:

$$\begin{bmatrix} -H(x,y) - D & A^T(x) \\ A(x) & E \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A^T(x) y \\ -h(x) + \mu Y^{-1} e \end{bmatrix}.$$

  Here, D and E are diagonal matrices involving slack variables,

$$H(x,y) = \nabla^2 f(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x) + \lambda I, \text{ and } A(x) = \nabla h(x),$$

  where λ is chosen to ensure appropriate descent properties.

- Compute step lengths to ensure positivity of slack variables.
- Shorten steps further to ensure a reduction in either infeasibility or in the barrier function.
- Step to new point, reduce barrier parameter (μ) and repeat.

Telescope Design

for Planet-Finding

Terrestrial Planet Finder Telescope (TPF)

- DETECT: Search 150-500 nearby (5-15 pc distant) Sun-like stars for Earth-like planets.
- CHARACTERIZE: Determine basic physical properties and measure "biomarkers", indicators of life or conditions suitable to support it.

Why Is It Hard? Can't Hubble do it?

- If the star is Sun-like and the planet is Earth-like, then the reflected visible light from the planet is 10^{-10} times as bright as the star. This is a difference of 25 magnitudes!
- If the star is 10 pc (33 ly) away and the planet is 1 AU from the star, the angular separation is 0.1 arcseconds!
- A point source (i.e. star) produces not a point image but an *Airy pattern* consisting of an *Airy disk* surrounded by a system of *diffraction rings* completely covering the nearby planet.
- By *apodizing* the entrance pupil, one can control the shape and strength of the diffraction rings.

Electric Field

The image-plane *electric field* $E()$ produced by an on-axis plane wave and an apodized aperture defined by an *apodization function* $A()$ is given by

$$\begin{aligned} E(\xi, \zeta) &= \iint_{\bigcirc} e^{i(x\xi + y\zeta)} A(x, y) dy dx \\ &\vdots \\ E(\rho) &= 2\pi \int_0^{1/2} J_0(r\rho) A(r) r dr, \end{aligned}$$

where J_0 denotes the 0-th order Bessel function of the first kind.

NOTE: The *electric field* depends *linearly* on the *apodization function*.

The unitless pupil-plane "length" r is given as a multiple of the aperture D.

The unitless image-plane "length" ρ is given as a multiple of focal-length times wavelength over aperture $(f\lambda/D)$ or, equivalently, as an angular measure on the sky, in which case it is a multiple of just λ/D. (Example: $\lambda = 0.5\mu$m and $D = 10$m implies $\lambda/D = 10$mas.)

The *intensity* is the square of the electric field.

Performance Metrics

Inner and Outer Working Angles

$$\rho_{\text{iwa}} \qquad \rho_{\text{owa}}$$

Contrast:

$$E^2(\rho)/E^2(0)$$

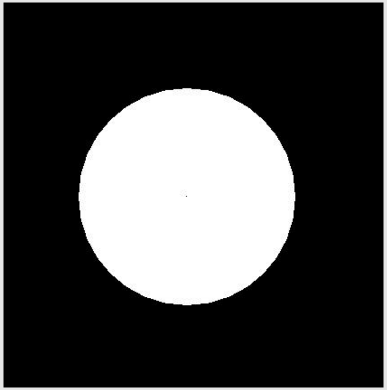
Airy Throughput:

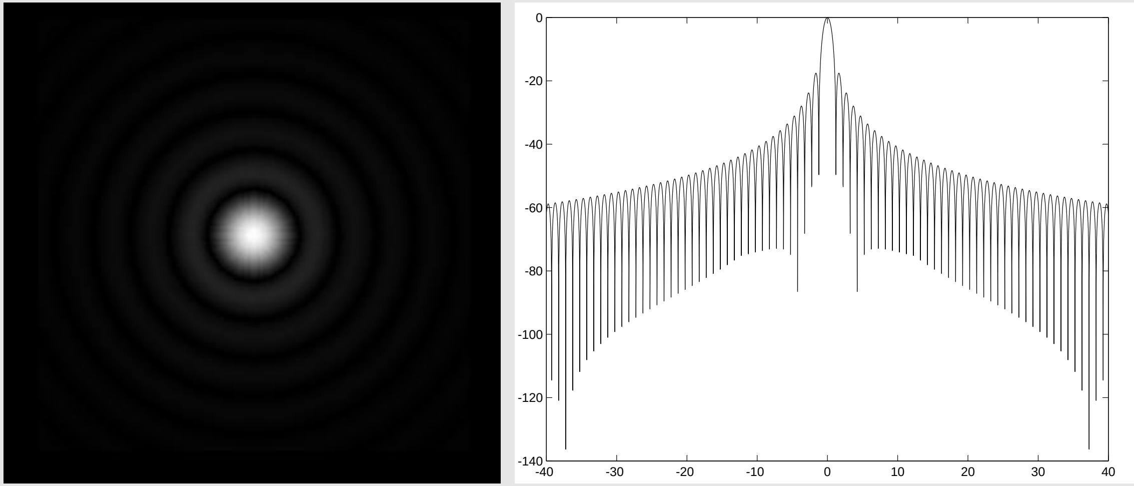
$$\frac{\displaystyle\int_0^{\rho_{\text{iwa}}} E^2(\rho)2\pi\rho d\rho}{(\pi(1/2)^2)} = 8\int_0^{\rho_{\text{iwa}}} E^2(\rho)\rho d\rho.$$

Clear Aperture—Airy Pattern

$\rho_{\text{iwa}} = 1.24 \quad \mathcal{T}_{\text{Airy}} = 84.2\% \quad \text{Contrast} = 10^{-2}$

$\rho_{\text{iwa}} = 748 \quad \mathcal{T}_{\text{Airy}} = 100\% \quad \text{Contrast} = 10^{-10}$





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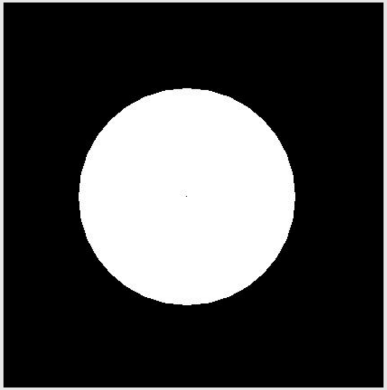
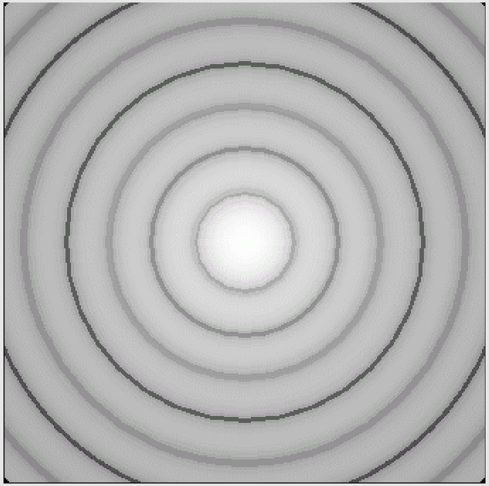
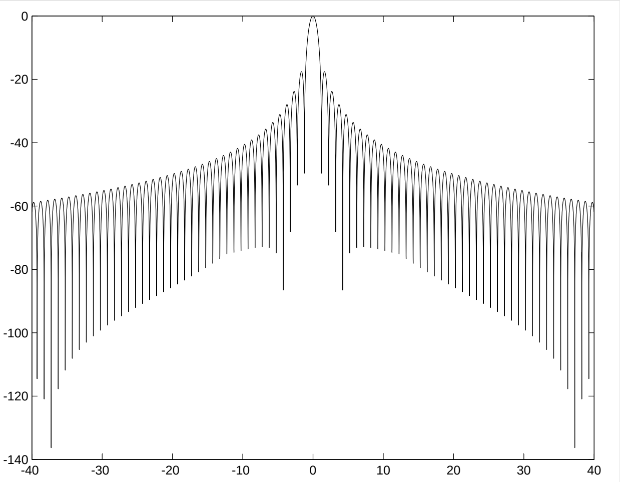






Optimization

Find *apodization* function $A()$ that solves:

$$\begin{array}{ll} \text{maximize} & \displaystyle\int_0^{1/2} A(r) 2\pi r dr \\ \text{subject to} & -10^{-5}E(0) \leq E(\rho) \leq 10^{-5}E(0), \qquad \rho_{\text{iwa}} \leq \rho \leq \rho_{\text{owa}}, \\ & \qquad 0 \leq A(r) \leq 1, \qquad\qquad 0 \leq r \leq 1/2, \end{array}$$

Optimization

Find *apodization* function $A()$ that solves:

$$\begin{array}{lll}
\text{maximize} & \displaystyle\int_0^{1/2} A(r) 2\pi r dr & \\
\text{subject to} & -10^{-5} E(0) \leq E(\rho) \leq 10^{-5} E(0), & \rho_{\text{iwa}} \leq \rho \leq \rho_{\text{owa}}, \\
& 0 \leq A(r) \leq 1, & 0 \leq r \leq 1/2, \\
& -50 \leq A''(r) \leq 50, & 0 \leq r \leq 1/2
\end{array}$$

An infinite dimensional *linear programming* problem.

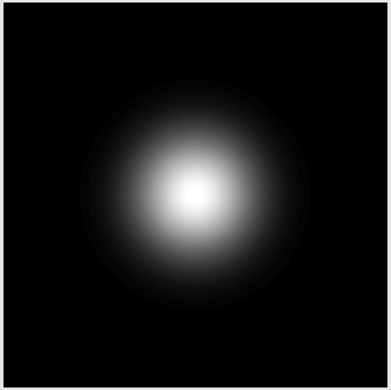
The AMPL Model

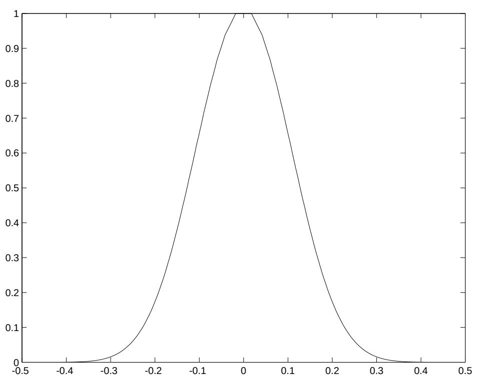
```
function J0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

param dr := (1/2)/N;
set Rs ordered := setof {j in 0.5..N-0.5 by 1} (1/2)*j/N;

var A {Rs} >= 0, <= 1, := 1/2;

set Rhos ordered := setof {j in 0..N} j*rho1/N;
set PlanetBand := setof {rho in Rhos: rho>=rho0 && rho<=rho1} rho;

var E0 {rho in Rhos} =
    2*pi*sum {r in Rs} A[r]*J0(2*pi*r*rho)*r*dr;

maximize area: sum {r in Rs} 2*pi*A[r]*r*dr;
subject to sidelobe_pos {rho in PlanetBand}: E0[rho] <= 10^(-5)*E0[0];
subject to sidelobe_neg {rho in PlanetBand}: -10^(-5)*E0[0] <= E0[rho];

subject to smooth {r in Rs: r != first(Rs) && r != last(Rs)}:
    -50*dr^2 <= A[next(r)] - 2*A[r] + A[prev(r)] <= 50*dr^2;

solve;
```

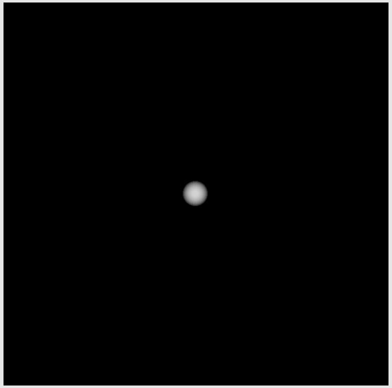
The Optimal Apodization

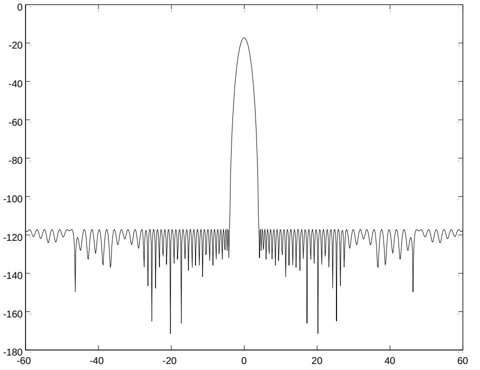
$$\rho_{\mathsf{iwa}} = 4 \qquad \mathcal{T}_{\mathsf{Airy}} = 9\%$$

Excellent dark zone. Unmanufacturable.









Concentric Ring Masks

Recall that for circularly symmetric apodizations

$$E(\rho) = 2\pi \int_0^{1/2} J_0(r\rho)A(r)rdr,$$

where J_0 denotes the 0-th order Bessel function of the first kind.

Let

$$A(r) = \begin{cases} 1 & r_{2j} \leq r \leq r_{2j+1}, \qquad j = 0, 1, \ldots, m-1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$0 \leq r_0 \leq r_1 \leq \cdots \leq r_{2m-1} \leq 1/2.$$

The integral can now be written as a sum of integrals and each of these integrals can be explicitly integrated to get:

$$E(\rho) \;=\; \sum_{j=0}^{m-1} \frac{1}{\rho} \left(r_{2j+1} J_1\left(\rho r_{2j+1}\right) - r_{2j} J_1\left(\rho r_{2j}\right) \right).$$

Mask Optimization Problem

$$\text{maximize} \sum_{j=0}^{m-1} \pi(r_{2j+1}^2 - r_{2j}^2)$$

$$\text{subject to:} \quad -10^{-5}E(0) \leq E(\rho) \leq 10^{-5}E(0), \quad \text{for } \rho_0 \leq \rho \leq \rho_1$$

where $E(\rho)$ is the function of the r_j's given on the previous slide.

This problem is a semiinfinite nonconvex optimization problem.

The AMPL Model

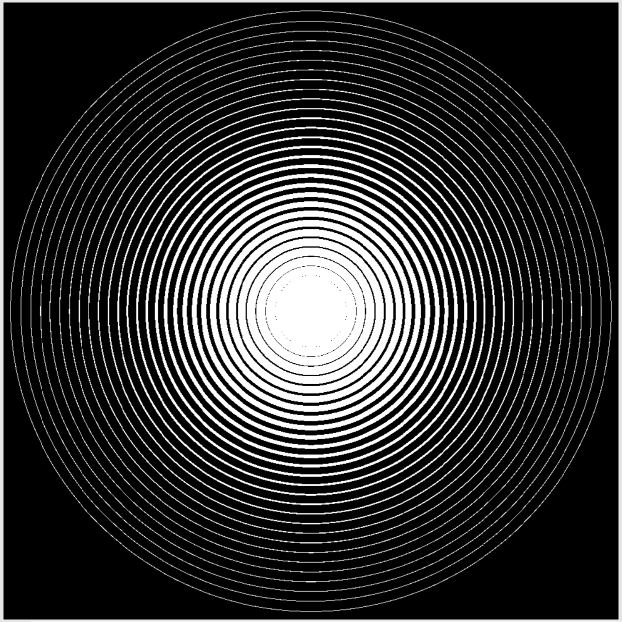
```
function intrJ0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

var r {j in 0..M} >= 0, <= 1/2, := r0[j];

set Rhos2 ordered := setof {j in 0..N} (j+0.5)*rho1/N;
set PlanetBand2 := setof {rho in Rhos2: rho>=rho0 && rho<=rho1} rho;

var E {rho in Rhos2} =
    (1/(2*pi*rho)^2)*sum {j in 0..M by 2}
(intrJ0(2*pi*rho*r[j+1]) - intrJ0(2*pi*rho*r[j]));

maximize area2: sum {j in 0..M by 2} (pi*r[j+1]^2 - pi*r[j]^2);
subject to sidelobe_pos2 {rho in PlanetBand2}: E[rho] <= 10^(-5)*E[first(rhos2)];
subject to sidelobe_neg2 {rho in PlanetBand2}: -10^(-5)*E[first(rhos2)] <= E[rho];

subject to order {j in 0..M-1}: r[j+1] >= r[j];

solve mask;
```

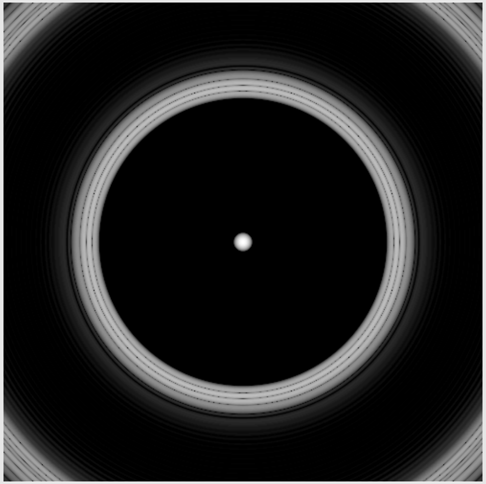
The Best Concentric Ring Mask

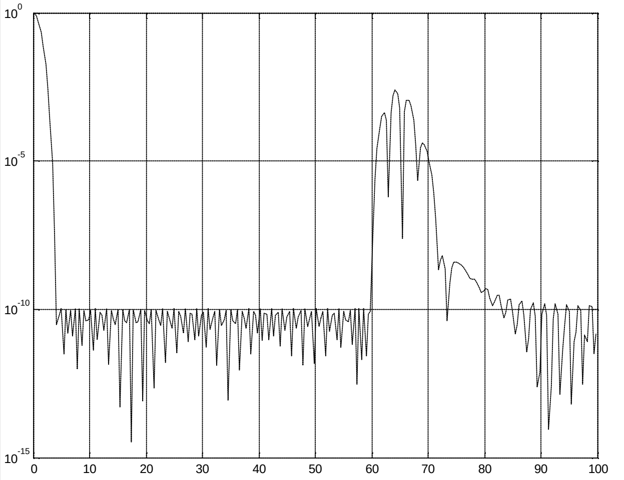
$\rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 60$

$\mathcal{T}_{\text{Airy}} = 9\%$

Lay it on glass?







Other Masks

Consider a binary apodization (i.e., a mask) consisting of an opening given by

$$A(x,y) = \begin{cases} 1 & |y| \leq a(x) \\ 0 & \text{else} \end{cases}$$

We only consider masks that are symmetric with respect to both the x and y axes. Hence, the function $a()$ is a nonnegative even function.

In such a situation, the electric field $E(\xi,\zeta)$ is given by

$$\begin{aligned} E(\xi,\zeta) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-a(x)}^{a(x)} e^{i(x\xi+y\zeta)} dy dx \\ &= 4 \int_{0}^{\frac{1}{2}} \cos(x\xi) \frac{\sin(a(x)\zeta)}{\zeta} dx \end{aligned}$$

Maximizing Throughput

Because of the symmetry, we only need to optimize in the first quadrant:

$$\text{maximize } 4\int_0^{\frac{1}{2}} a(x)dx$$

$$\begin{array}{lll} \text{subject to} & -10^{-5}E(0,0) \le E(\xi,\zeta) \le 10^{-5}E(0,0), & \text{for } (\xi,\zeta) \in \mathcal{O} \\ & 0 \le a(x) \le 1/2, & \text{for } 0 \le x \le 1/2 \end{array}$$

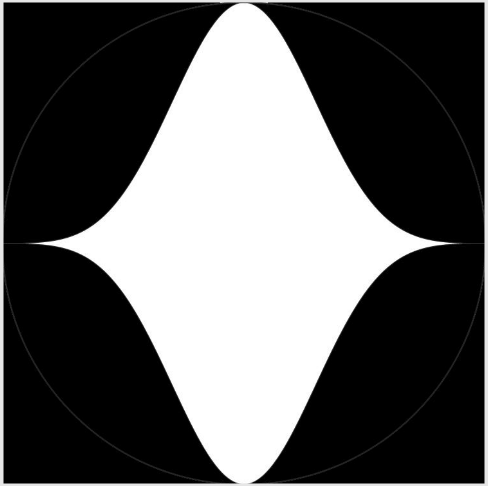
The objective function is the total open area of the mask. The first constraint guarantees 10^{-10} light intensity throughout a desired region of the focal plane, and the remaining constraint ensures that the mask is really a mask.

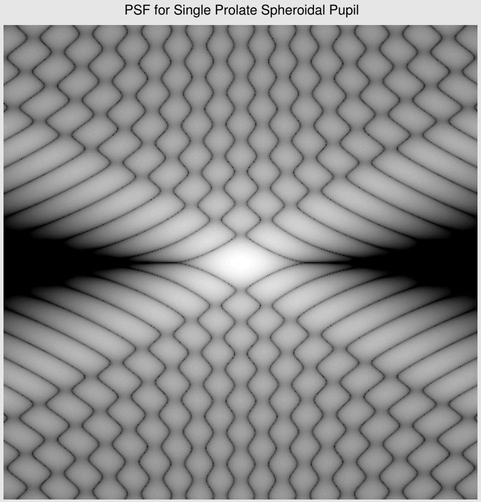
If the set $\mathcal{O}$ is a subset of the x-axis, then the problem is an infinite dimensional linear programming problem.

One Pupil w/ On-Axis Constraints

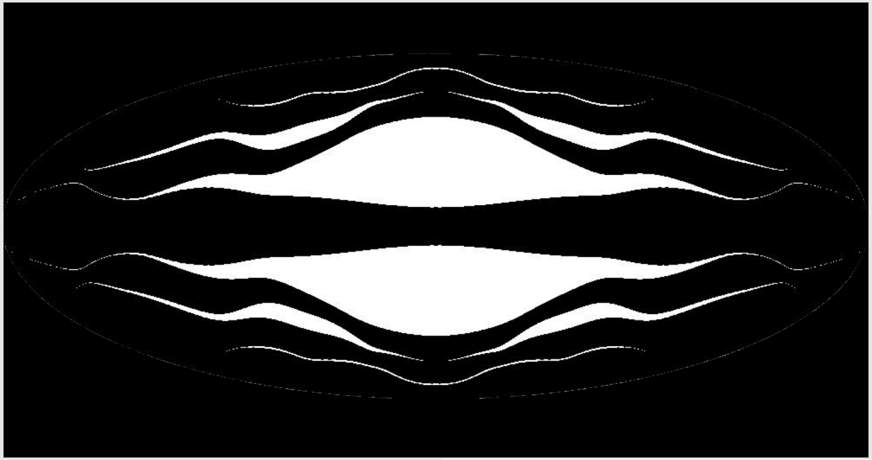
$$\rho_{\mathrm{iwa}} = 4 \quad \mathcal{T}_{\mathrm{Airy}} = 43\%$$

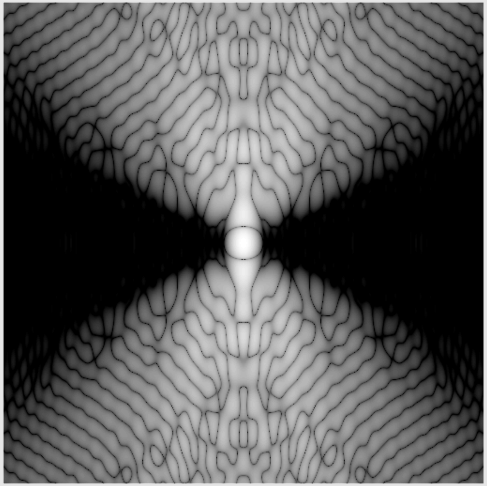
Small dark zone...Many rotations required




PSF for Single Prolate Spheroidal Pupil

Multiple Pupil Mask





$$\rho_{\mathsf{iwa}} = 4$$

$$\mathcal{T}_{\mathsf{Airy}} = 30\%$$

Throughput relative to ellipse
11% central obstr.
Easy to make
Only a few rotations

Space Occulter Design

for Planet-Finding

Space-based Occulter (TPF-O)

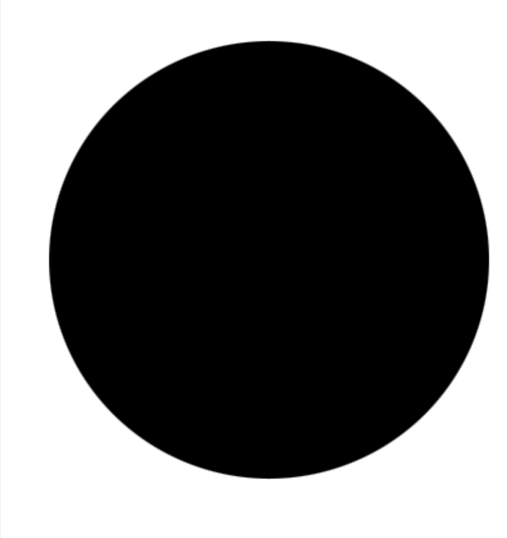
Telescope Aperture: 4m, Occulter Diameter: 50m, Occulter Distance: $72,000$km

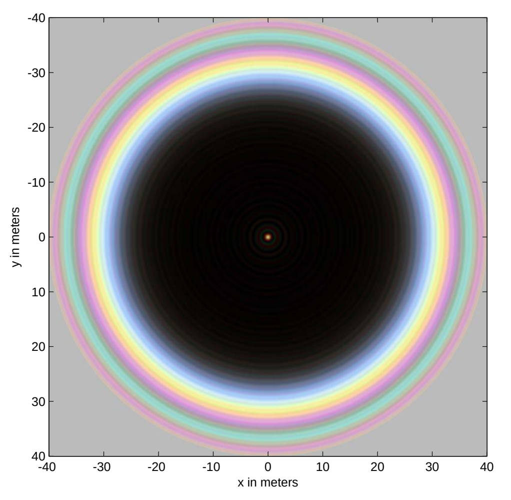
Plain External Occulter (Doesn't Work!)

Shadow ⇒

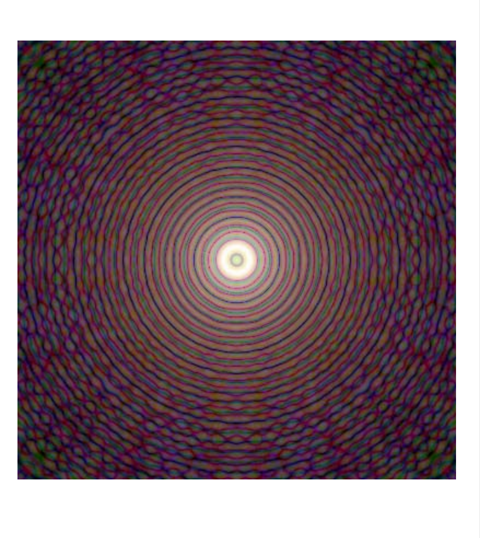
Circular Occulter

⇓



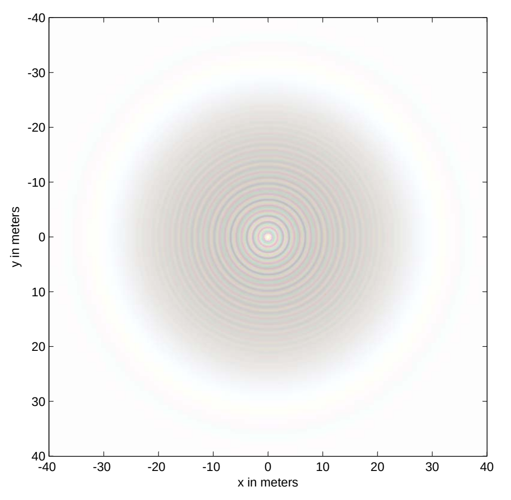


⇐ Note bright spot at center
(Poisson's spot)



⇑

Telescope Image

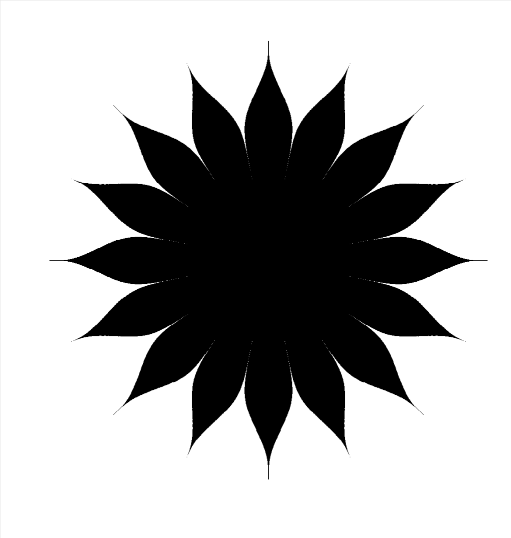


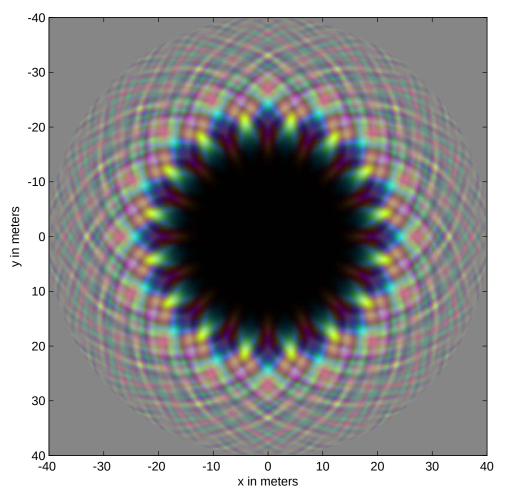
⇐ Shadow (Log Stretch)

Shaped Occulter—Eliminates Poisson's Spot

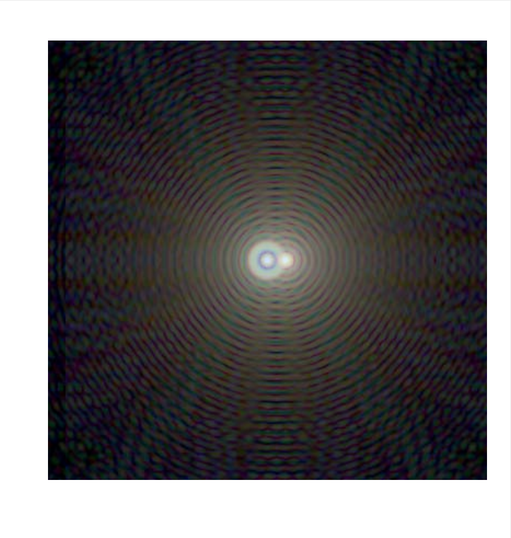
Shadow ⇒

Shaped Occulter
⇓

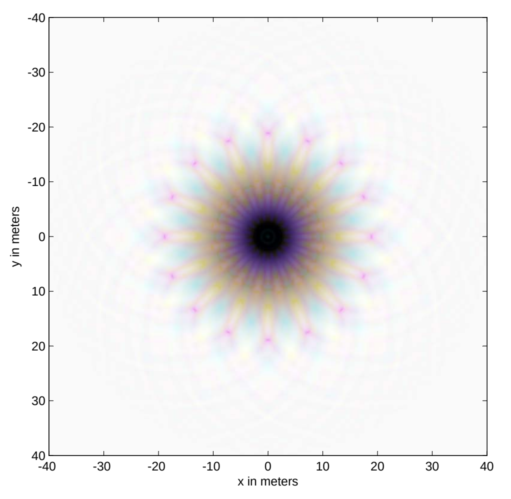




⇐ Bright spot is gone



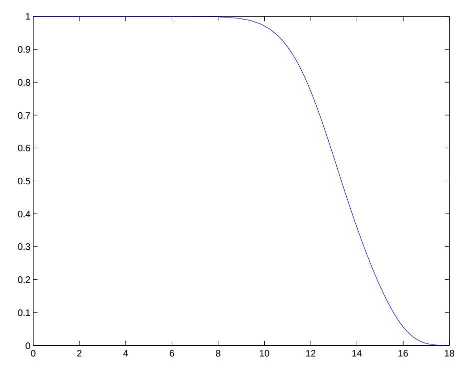
⇑
Telescope image shows planet



⇐ Shadow is dark
(Log Stretch)

Apodized Occulters

Apodized Occulter



Radial Attenuation $A(r)$

- The problem (as before) is *diffraction*.
- Abrupt edges create unwanted diffraction.
- *Solution:* Soften the edges with a partially transmitting material—an *apodizer*.
- Let $A(r, \theta)$ denote *attenuation* at location (r, θ) on the occulter.
- The *intensity* of the downstream light is given by the *square of the magnitude of the electric field* $E(\rho, \phi)$.
- *Babinet's principle* plus *Fresnel propagation* gives a formula for the downstream electric field:

$$E(\rho, \phi) = 1 - \frac{1}{i\lambda z} \int_0^\infty \int_0^{2\pi} e^{\frac{i\pi}{\lambda z}(r^2+\rho^2-2r\rho\cos(\theta-\phi))} A(r, \theta) r \, d\theta \, dr.$$

where

  - z is distance “downstream” and
  - λ is wavelength of light.

Attenuation Profile Optimization

$$
\begin{array}{lrll}
\text{minimize} & & \gamma & \\
\text{subject to} & -\gamma & \leq \Re(E(\rho)) \leq \gamma & \text{for } \rho \in \mathcal{R}, \quad \lambda \in \Lambda \\
& -\gamma & \leq \Im(E(\rho)) \leq \gamma & \text{for } \rho \in \mathcal{R}, \quad \lambda \in \Lambda \\
& & A'(r) \leq 0 & \text{for } 0 \leq r \leq R \\
& -d & \leq A''(r) \leq d & \text{for } 0 \leq r \leq R
\end{array}
$$

Specific choice:

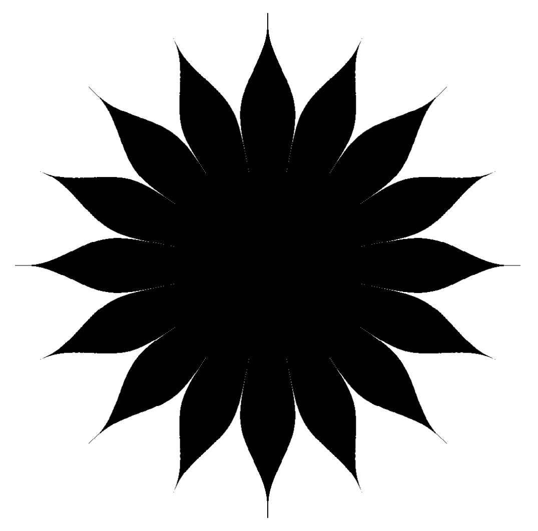
$$R = 25, \quad d = 0.04, \quad \mathcal{R} = [0, 3], \quad \Lambda = [0.4, 1.1] \times 10^{-6}$$

where all metric quantities are in meters.

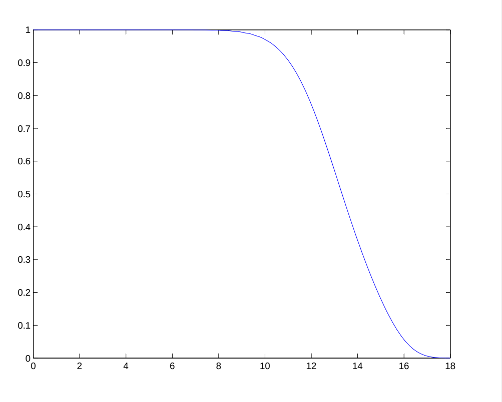
An infinite dimensional linear programming problem.
Discretize:

- $[0, R]$ into 5000 evenly space points.
- $\mathcal{R}$ into 150 evenly spaced points.
- Λ into increments of 0.1×10^{-6}.

Petal-Shaped Occulters



16-Petal Occulter $A(r,\theta)$



Radial Attenuation $A(r)$

- From Jacobi-Anger expansion we get:

$$\begin{aligned} E(\rho,\phi) &= 1 - \frac{2\pi}{i\lambda z}\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_0\left(\frac{2\pi r\rho}{\lambda z}\right) A(r) r dr \\ &\quad - \sum_{k=1}^{\infty} \frac{2\pi(-1)^k}{i\lambda z}\left(\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_{kN}\left(\frac{2\pi r\rho}{\lambda z}\right)\frac{\sin(\pi k A(r))}{\pi k} r dr\right) \\ &\quad \times \left(2\cos(kN(\phi - \frac{\pi}{2}))\right) \end{aligned}$$

  where N is the number of petals.

- For small ρ, truncated summation well-approximates full sum.
- Truncated after 10 terms.
- $\lambda \in [0.4, 1.1]$ microns.
- $z = 72,000$ km, $R = 25$ m.
- In angular terms, $R/z = 73$ mas.

Celestial Mechanics: The N-Body Problem

Least Action Principle

Given: n bodies.

Let:

m_j denote the mass and
$z_j(t)$ denote the position in $\mathbb{R}^2 = \mathbb{C}$ of body j at time t.

Action Functional:

$$A = \int_0^{2\pi} \left(\sum_j \frac{m_j}{2} \|\dot{z}_j\|^2 + \sum_{j,k:k<j} \frac{m_j m_k}{\|z_j - z_k\|} \right) dt.$$

Critical points satisfy equations of motion.

Minimize!

Equations of Motion

First Variation:

$$\begin{aligned}\delta A &= \int_0^{2\pi} \sum_\alpha \left(\sum_j m_j \dot{z}_j^\alpha \delta\dot{z}_j^\alpha - \sum_{j,k:k<j} m_j m_k \frac{(z_j^\alpha - z_k^\alpha)(\delta z_j^\alpha - \delta z_k^\alpha)}{\|z_j - z_k\|^3} \right) dt \\ &= -\int_0^{2\pi} \sum_j \sum_\alpha \left(m_j \ddot{z}_j^\alpha + \sum_{k:k\neq j} m_j m_k \frac{z_j^\alpha - z_k^\alpha}{\|z_j - z_k\|^3} \right) \delta z_j^\alpha dt\end{aligned}$$

Setting first variation to zero, we get:

$$m_j \ddot{z}_j^\alpha = -\sum_{k:k\neq j} m_j m_k \frac{z_j^\alpha - z_k^\alpha}{\|z_j - z_k\|^3}, \qquad j = 1, 2, \ldots, n, \quad \alpha = 1, 2$$

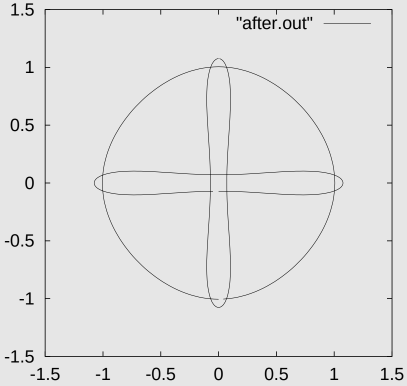
Note: If $m_j = 0$ for some j, then the first order optimality condition reduces to $0 = 0$, which is *not* the equation of motion for a massless body.

The AMPL Model. Nonconvex!!

```
param N := 3;  # number of masses
param n := 15;  # number of terms in Fourier series representation
param m := 100;  # number of terms in numerical approx to integral

param theta {j in 0..m-1} := j*2*pi/m;

var x {i in 0..N-1, j in 0..m-1};
var y {i in 0..N-1, j in 0..m-1};

var xdot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (x[i,j+1]-x[i,j])*m/(2*pi) else (x[i,0]-x[i,m-1])*m/(2*pi);
var ydot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (y[i,j+1]-y[i,j])*m/(2*pi) else (y[i,0]-y[i,m-1])*m/(2*pi);

var KineticEnergy {j in 0..m-1} = 0.5*sum {i in 0..N-1} (xdot[i,j]^2 + ydot[i,j]^2);

var PotentialEnergy {j in 0..m-1}
  = - sum {i in 0..N-1, ii in 0..N-1: ii>i}
            1/sqrt((x[i,j]-x[ii,j])^2 + (y[i,j]-y[ii,j])^2);

minimize A: (2*pi/m)*sum {j in 0..m-1} (KineticEnergy[j] - PotentialEnergy[j]);
```

Choreographies and the Ducati

The previous AMPL model was used to find many *choreographies* (a la Moore and Montgomery/Chencinier) in the equimass n-body problem and the stable *Ducati* solution to the 3-body problem.



`http://www.princeton.edu/~rvdb/JAVA/astro/galaxy/Galaxy.html`