Making Dark Shadows with Linear Programming

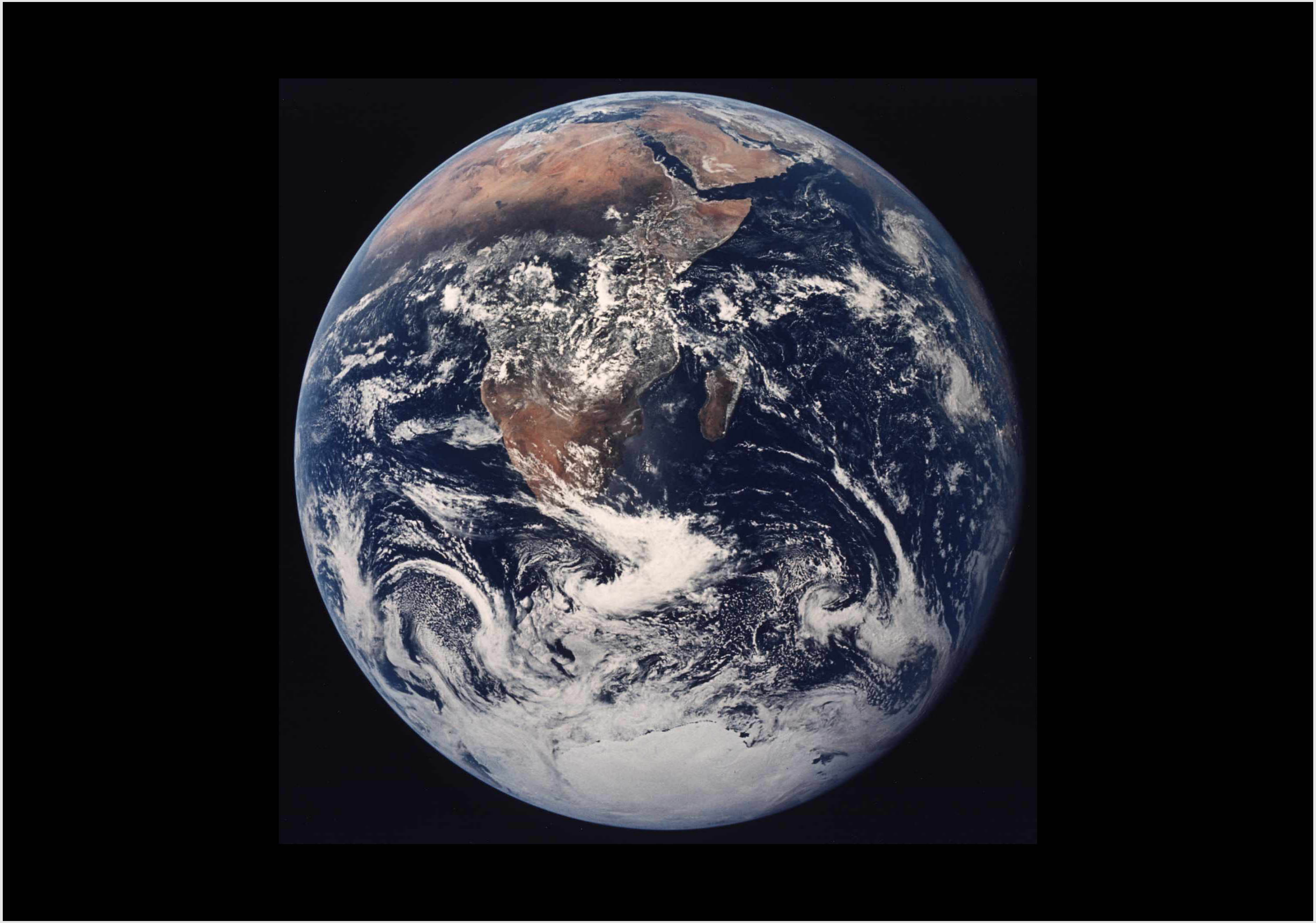
Robert J. Vanderbei

2008 October 13

INFORMS 2008
Washington DC

http://www.princeton.edu/~rvdb

Are We Alone?



Indirect Detection Methods

Over 300 planets found—more all the time

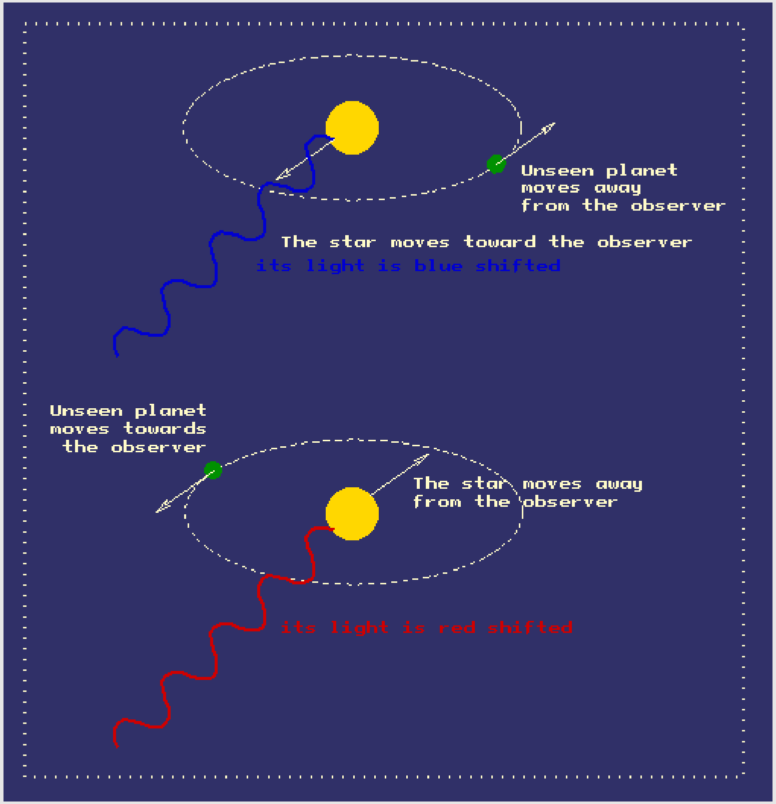
Wobble Methods

Radial Velocity.

For edge-on systems.
Measure periodic doppler shift.

Astrometry.

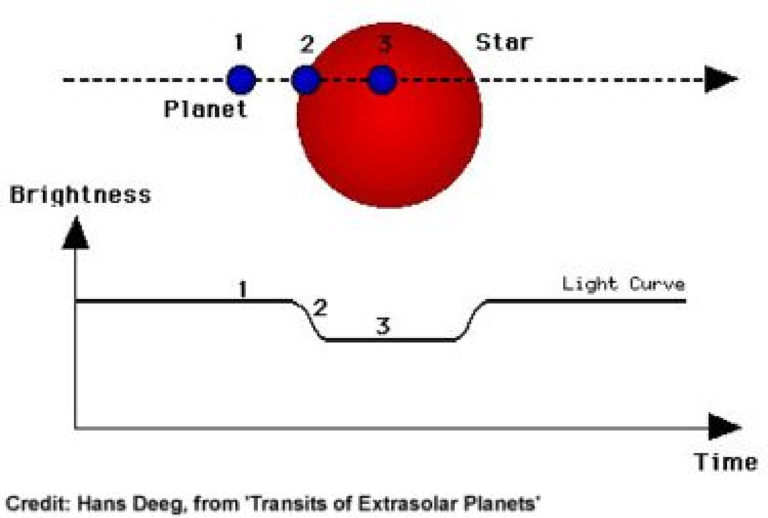
Best for face-on systems.
Measure circular wobble against background stars.



Transit Method

- HD209458b confirmed both via RV and transit.
- Period: 3.5 days
- Separation: 0.045 AU (0.001 arcsecs)
- Radius: $1.3 R_J$
- Intensity Dip: $\sim 1.7\%$
- Venus Dip $= 0.01\%$, Jupiter Dip: 1%

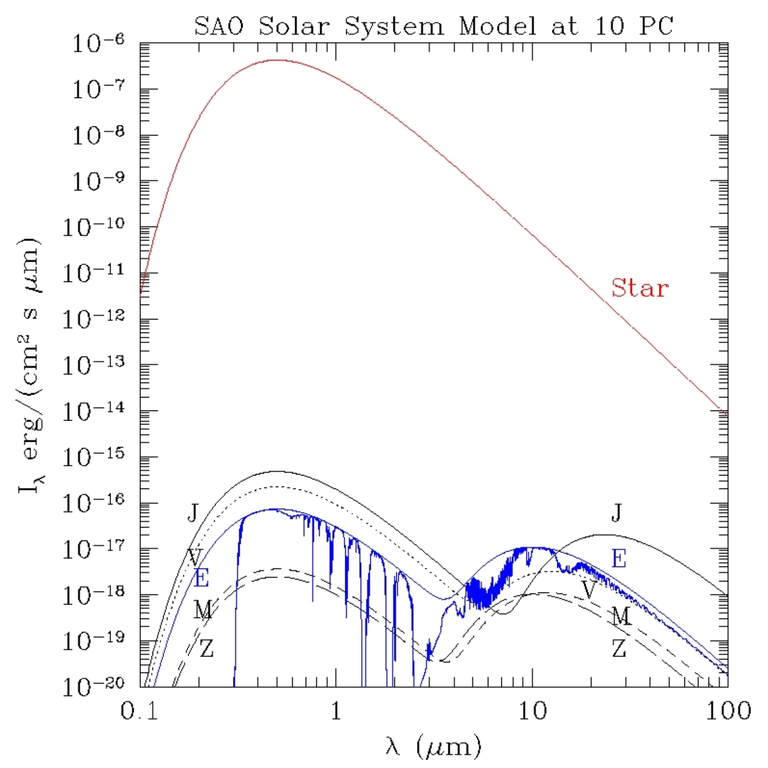
Venus Transit (R.J. Vanderbei)



Direct Detection

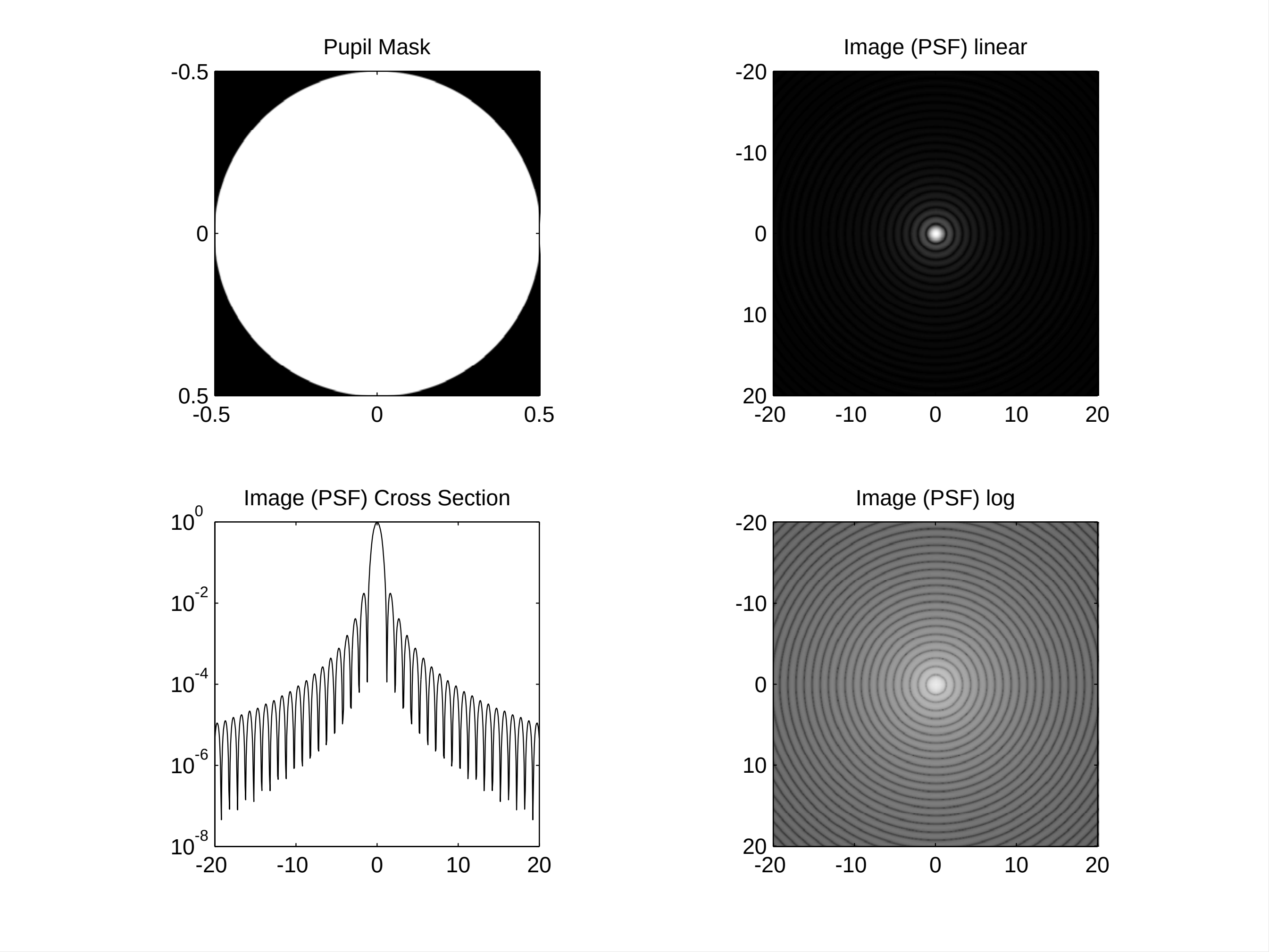
Why It's Hard

- *Bright Star/Faint Planet:* In visible light, our Sun is 10^{10} times brighter than Earth. That's 25 mags.
- *Close to Each Other:* A planet at 1 AU from a star at 10 parsecs can appear at most 0.1 arcseconds in separation.
- *Far from Us:* There are less than 100 Sun-like stars within 10 parsecs.



Telescope w/ Unobstructed Aperture

Doesn't Work! Requires an aperture measured in kilometers to mitigate diffraction effects.

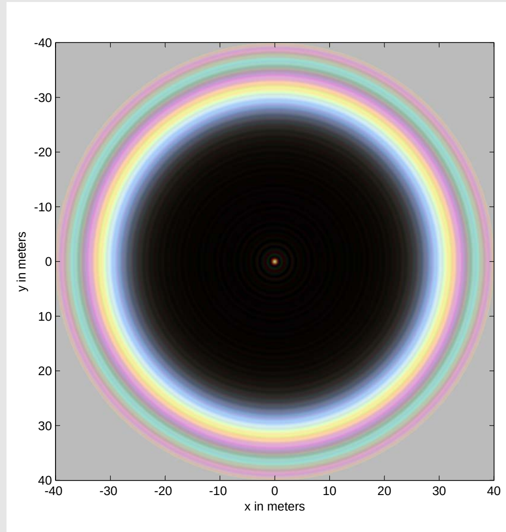


Space-based Occulter (TPF-O)

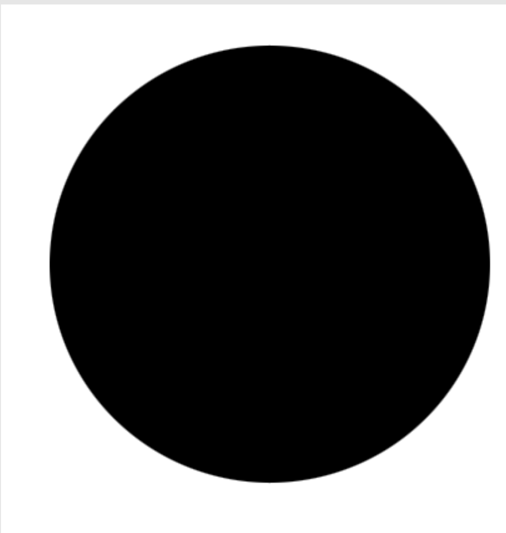
Telescope Aperture: 4m, Occulter Diameter: 50m, Occulter Distance: $72,000$km

Plain External Occulter (Doesn't Work!)

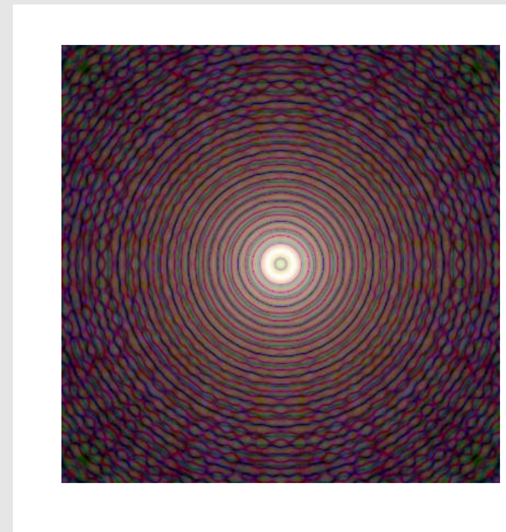
Shadow ⇒



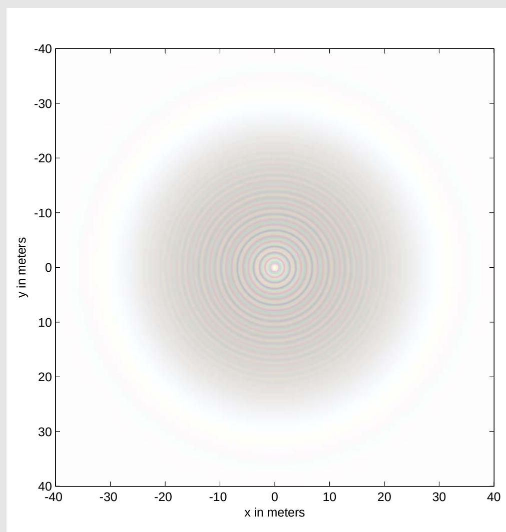
Circular Occulter
⇓



⇐ Note bright spot at center (Poisson's spot)

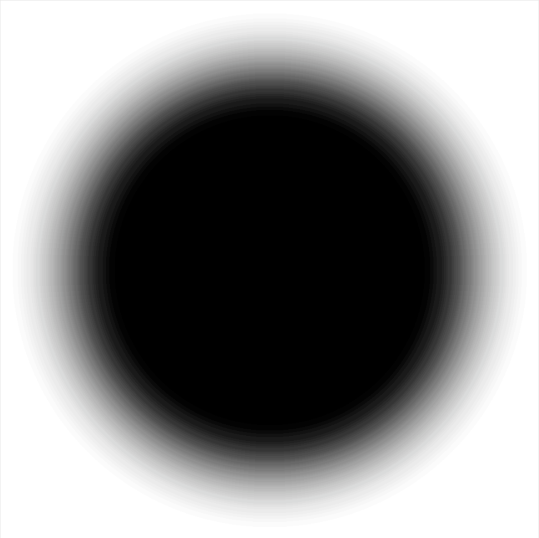


⇑
Telescope Image

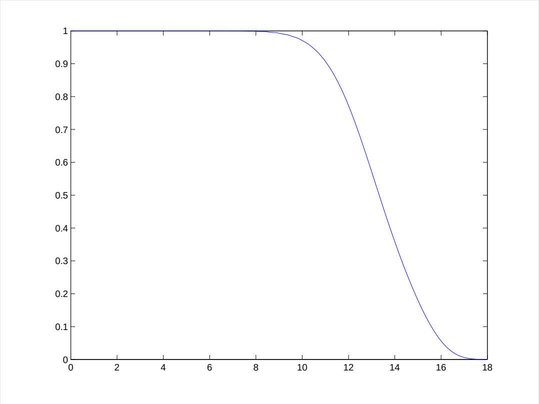


⇐ Shadow (Log Stretch)

Apodized Occulters



Apodized Occulter



Radial Attenuation $A(r)$

- The problem is *diffraction*.
- Abrupt edges create unwanted diffraction.
- *Solution:* Soften the edges with a partially transmitting material—an *apodizer*.
- Let $A(r, \theta)$ denote *attenuation* at location (r, θ) on the occulter.
- The *intensity* of the downstream light is given by the *square of the magnitude of the electric field* $E(\rho, \phi)$.
- *Babinet's principle* plus *Fresnel propagation* gives a formula for the downstream electric field:

$$E(\rho, \phi) = 1 - \frac{1}{i\lambda z} \int_0^\infty \int_0^{2\pi} e^{\frac{i\pi}{\lambda z}(r^2+\rho^2-2r\rho\cos(\theta-\phi))} A(r, \theta) r d\theta dr.$$

where

- z is distance "downstream" and
- λ is wavelength of light.

Attenuation Profile Optimization

$$
\begin{array}{llll}
\text{minimize} & \gamma & & \\
\text{subject to} & -\gamma \le \Re(E(\rho)) \le \gamma & & \text{for } \rho \in \mathcal{R}, \lambda \in \mathcal{L} \\
 & -\gamma \le \Im(E(\rho)) \le \gamma & & \text{for } \rho \in \mathcal{R}, \lambda \in \mathcal{L} \\
 & A'(r) \le 0 & & \text{for } 0 \le r \le R \\
 & -d \le A''(r) \le d & & \text{for } 0 \le r \le R
\end{array}
$$

Specific choice:

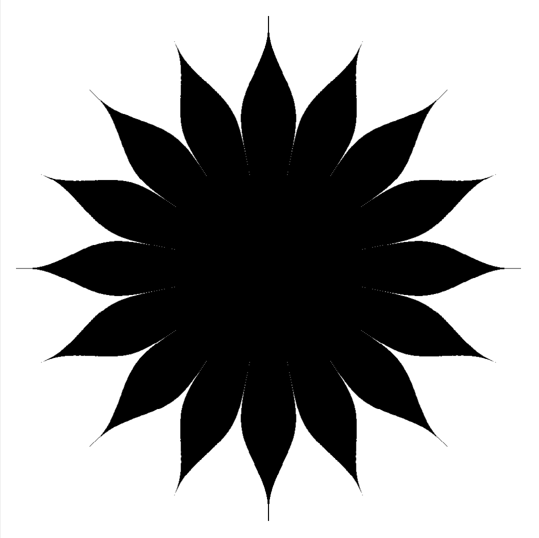
$$R = 25, \quad d = 0.04, \quad \mathcal{R} = [0, 3], \quad \mathcal{L} = [0.4, 1.1] \times 10^{-6}$$

where all metric quantities are in meters.

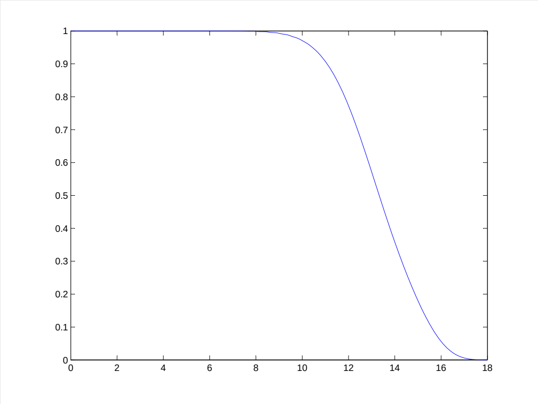
An infinite dimensional linear programming problem.
Discretize:

- $[0, R]$ into 5000 evenly space points.
- $\mathcal{R}$ into 150 evenly spaced points.
- $\mathcal{L}$ into increments of 0.1×10^{-6}.

Petal-Shaped Occulters



16-Petal Occulter $A(r,\theta)$



Radial Attenuation $A(r)$

- From Jacobi-Anger expansion we get:

$$\begin{aligned} E(\rho,\phi) &= 1 - \frac{2\pi}{i\lambda z}\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_0\left(\frac{2\pi r\rho}{\lambda z}\right) A(r) r dr \\ &\quad - \sum_{k=1}^{\infty} \frac{2\pi(-1)^k}{i\lambda z}\left(\int_0^R e^{\frac{i\pi}{\lambda z}(r^2+\rho^2)} J_{kN}\left(\frac{2\pi r\rho}{\lambda z}\right)\frac{\sin(\pi k A(r))}{\pi k} r dr\right) \\ &\quad \times \left(2\cos(kN(\phi - \frac{\pi}{2}))\right) \end{aligned}$$

  where N is the number of petals.

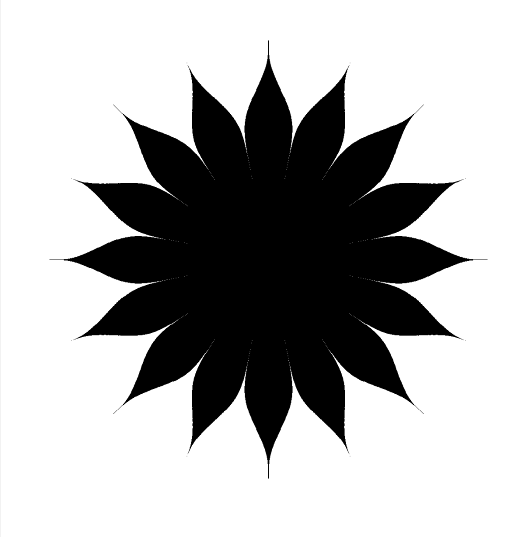
- For small ρ, truncated summation well-approximates full sum.
- Truncated after 10 terms.
- $\lambda \in [0.4, 1.1]$ microns.
- $z = 72,000$ km, $R = 25$ m.
- In angular terms, $R/z = 0.073$ arcseconds.

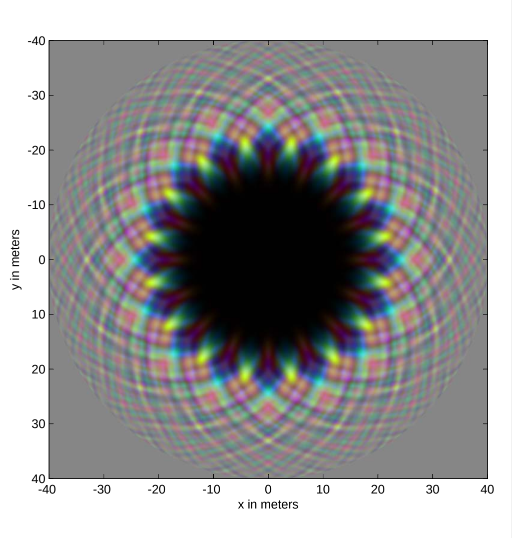
Shaped Occulter

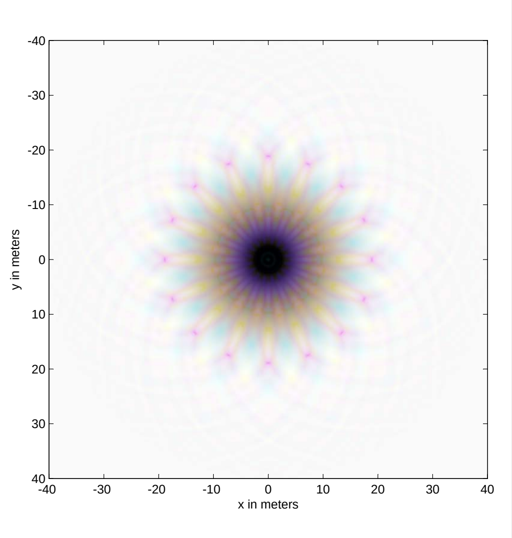
Shadow ⇒

Shaped Occulter

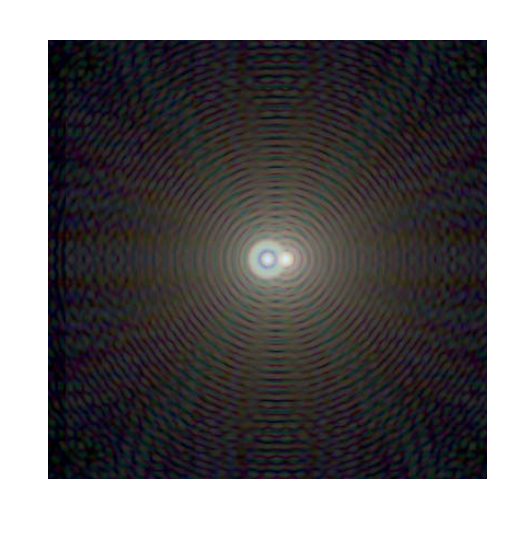
⇓







⇐ Bright spot is gone



⇑

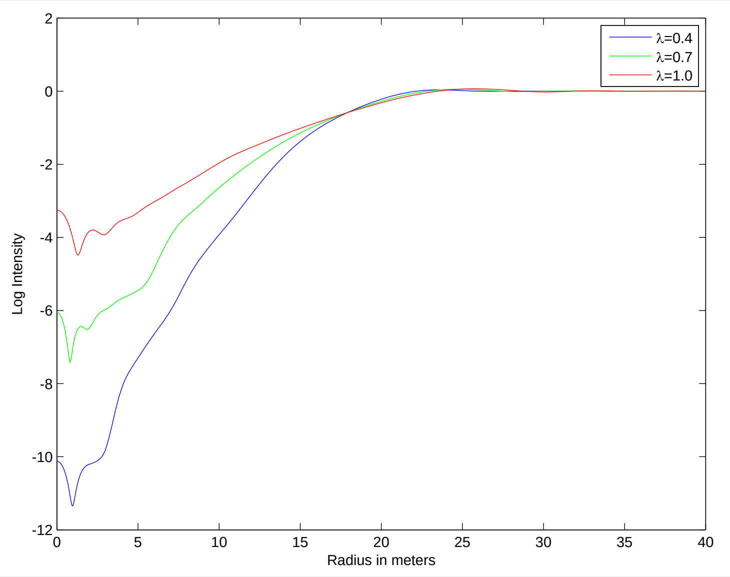
Telescope image shows planet

⇐ Shadow is dark (Log Stretch)

Sub-Optimal Hypergaussian

Hypergaussian profile:

$$A(r) = \begin{cases} 1 & r < 9.5 \\ e^{-(r/9.5-1)^6} & 9.5 \leq r < 25 \\ 0 & 25 \leq r. \end{cases}$$



Need to increase size and distance. New distance: $158,400$ km.

AMPL Model—Data/Constants

```
function J0;

param pi := 4*atan(1);
param pi2 := pi/2;
param N := 4000;        # discretization parameter at occulter plane
param M := 150;         # discretization parameter at telescope plane
param c := 25.0;        # overall radius of occulter
param z := 72000e+3;    # distance from telescope to apodized occulter
param lambda {3..11};   # set of wavelengths
param rho1 := 25;       # max radius investigated at telescope's pupil plane
param rho_end := 3;     # radius below which high contrast is required

# a few convenient shorthands
param lz {j in 3..11 by 1.0} := lambda[j]*z;
param pi2lz {j in 3..11 by 1.0} := 2*pi/lz[j];
param pilz {j in 3..11 by 1.0} := pi/lz[j];

param dr := c/N;
set Rs ordered;
let Rs := setof {j in 1..N by 1} c*(j-0.5)/N;

set Rhos ordered;
let Rhos := setof {j in 0..M} (j/M)*rho1;
```

AMPL Model—Model

```
function J0;

var A {r in Rs} >= 0, <= 1;
var contrast >= 0;
var Ereal {j in 3..11 by 1.0, rho in Rhos} =
     1-pi2lz[j]*
     sum {r in Rs} sin(pilz[j]*(r^2+rho^2))*A[r]*J0(-pi2lz[j]*r*rho)*r*dr;
var Eimag {j in 3..11 by 1.0, rho in Rhos} =
    pi2lz[j]*
    sum {r in Rs} cos(pilz[j]*(r^2+rho^2))*A[r]*J0(-pi2lz[j]*r*rho)*r*dr;

minimize cont: contrast;

subject to main_real_neg {j in 3..11 by 1.0, rho in Rhos: rho < rho_end}:
    -contrast <= Ereal[j,rho];
subject to main_real_pos {j in 3..11 by 1.0, rho in Rhos: rho < rho_end}:
    Ereal[j,rho] <= contrast;
subject to main_imag_neg {j in 3..11 by 1.0, rho in Rhos: rho < rho_end}:
    -contrast <= Eimag[j,rho];
subject to main_imag_pos {j in 3..11 by 1.0, rho in Rhos: rho < rho_end}:
    Eimag[j,rho] <= contrast;

subject to monotone {r in Rs: r>first(Rs)}: A[prev(r)] >= A[r];
subject to smooth   {r in Rs: r>first(Rs) && r<last(Rs)}:
    -0.044 <= (A[next(r)]-2*A[r]+A[prev(r)])/dr^2 <= 0.044;

let {j in 3..11 by 1.0} lambda[j] := (j-0.5)*1e-7;

solve;
```