Reconnect 2022 – Optimization

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DIMACS
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https://vanderbei.princeton.edu/DIMACS/DIMACS_Reconnect.pdf

Outline

- Theory/Algorithms
  - Linear Programming
    - Simplex Method w/ Pivot Tools
      - Pivoting to Optimality
      - Unboundedness
      - Degeneracy
      - Efficiency
      - Duality
      - Parametric Simplex Method
  - Integer Programming
    - Gomory Cuts
- Applications
  - Fair Grading
  - Financial Portfolio Optimization

Pivot Tools

- Simple Pivot Tool:
  vanderbei.princeton.edu/JAVA/pivot/simple.html
- Advanced Pivot Tool:
  vanderbei.princeton.edu/JAVA/pivot/advanced.html
- Cycling Example:
  vanderbei.princeton.edu/JAVA/pivot/cycle.html
- Exponentially Slow Example:
  vanderbei.princeton.edu/JAVA/pivot/kleeminty_simple.html
- Gomory Cuts:
  vanderbei.princeton.edu/JAVA/pivot/gomory.html
- Network Simplex:
  vanderbei.princeton.edu/JAVA/network/nettool/netsimp.html
- Transportation Problem:
  vanderbei.princeton.edu/JAVA/network/nettool/transportation.html
- Shortest Paths:
  vanderbei.princeton.edu/JAVA/network/shortpaths/shortpaths.html

Linear Programming (LP)

Standard Form.

$$
\begin{array}{lrcrcrcl}
\text{maximize} & c_1x_1 & + & c_2x_2 & + \cdots + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & \leq & b_2 \\
 & & & & \vdots & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & \leq & b_m \\
 & & & & x_1,\ x_2, & \ldots\ x_n & \geq & 0.
\end{array}
$$

Why it's hard:

- Lots of variables (n of 'em).
- Lots of "boundaries" to check (the inequalities).

Why it's not impossible:

- All expressions are linear.

Simplex Method for LP

An Example.

$$
\begin{array}{lrcl}
\text{maximize} & 5\,x_1 + 4\,x_2 + 3\,x_3 & & \\
\text{subject to} & 2\,x_1 + 3\,x_2 + \,x_3 & \leq & 5 \\
 & 4\,x_1 + \,x_2 + 2\,x_3 & \leq & 11 \\
 & 3\,x_1 + 4\,x_2 + 2\,x_3 & \leq & 8 \\
 & x_1,\ x_2,\ x_3 & \geq & 0.
\end{array}
$$

Rewrite with Slack Variables

$$
\begin{array}{ll}
\text{maximize} & 5\,x_1 + 4\,x_2 + 3\,x_3 \\
\text{subject to} & 2\,x_1 + 3\,x_2 + x_3 \le 5 \\
& 4\,x_1 + x_2 + 2\,x_3 \le 11 \\
& 3\,x_1 + 4\,x_2 + 2\,x_3 \le 8 \\
& x_1,\ x_2,\ x_3 \ge 0.
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{lrcrl}
\text{maximize} & \zeta & = & & + 5\,x_1 + 4\,x_2 + 3\,x_3 \\
\text{subject to} & w_1 & = & 5 & - 2\,x_1 - 3\,x_2 - x_3 \\
& w_2 & = & 11 & - 4\,x_1 - x_2 - 2\,x_3 \\
& w_3 & = & 8 & - 3\,x_1 - 4\,x_2 - 2\,x_3 \\
& \multicolumn{4}{r}{x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \ge 0.}
\end{array}
$$

Dictionary

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & + & 5x_1 & + & 4x_2 & + & 3x_3 \\
\text{subject to} & w_1 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
& w_2 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
& w_3 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
& & & & & \multicolumn{5}{l}{x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \ \geq \ 0.}
\end{array}
$$

Notes:

- This layout is called a *dictionary*: the variables on the left are "defined" in terms of the variables on the right.
- We will use the Greek letter ζ for the *objective function*.
- Dependent variables, on the left, are called *basic variables*.
- Independent variables, on the right, are called *nonbasic variables*.
- Setting x_1, x_2, and x_3 to 0, we can read off the values for the other variables: $w_1 = 5$, $w_2 = 11$, $w_3 = 8$. This specific "solution" is called a *basic solution* (aka *dictionary solution*).
  It's called a solution because it is one of many solutions to the system of linear equations. We are not implying that it is a solution to the optimization problem. We will call that the *optimal solution*.

Basic Solution is Feasible

We got lucky!

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad w_1 = 5, \quad w_2 = 11, \quad w_3 = 8$$

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & + & 5x_1 & + & 4x_2 & + & 3x_3 \\
\text{subject to} & w_1 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
& w_2 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
& w_3 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
& & & & & & & & & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0.
\end{array}$$

Notes:

- All the variables in the current basic solution are nonnegative.
- Such a solution is called *feasible*.
- The initial basic solution need not be feasible—we were just lucky above.

Simplex Method—First Iteration

$$
\begin{array}{rcrrrrrrr}
\zeta & = & & + & 5\,x_1 & + & 4\,x_2 & + & 3\,x_3 \\
\hline
w_1 & = & 5 & - & 2\,x_1 & - & 3\,x_2 & - & x_3 \\
w_2 & = & 11 & - & 4\,x_1 & - & x_2 & - & 2\,x_3 \\
w_3 & = & 8 & - & 3\,x_1 & - & 4\,x_2 & - & 2\,x_3 .
\end{array}
$$

- If x_1 increases, the objective ζ goes *up*.
- How much can x_1 increase? Until w_1 decreases to zero.
- Do it. End result: $x_1 > 0$ whereas $w_1 = 0$.
- That is, x_1 must become *basic* and w_1 must become *nonbasic*.
- Algebraically rearrange equations to, in the words of Jean-Luc Picard, "Make it so."
- This is a *pivot*.

A Pivot: $x_1 \leftrightarrow w_1$

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & + & 5\,x_1 & + & 4\,x_2 & + & 3\,x_3 \\
\hline
w_1 & = & 5 & - & 2\,x_1 & - & 3\,x_2 & - & x_3 \\
w_2 & = & 11 & - & 4\,x_1 & - & x_2 & - & 2\,x_3 \\
w_3 & = & 8 & - & 3\,x_1 & - & 4\,x_2 & - & 2\,x_3 .
\end{array}
$$

becomes

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 12.5 & - & 2.5\,w_1 & - & 3.5\,x_2 & + & 0.5\,x_3 \\
\hline
x_1 & = & 2.5 & - & 0.5\,w_1 & - & 1.5\,x_2 & - & 0.5\,x_3 \\
w_2 & = & 1.0 & + & 2\,w_1 & + & 5\,x_2 & & \\
w_3 & = & 0.5 & + & 1.5\,w_1 & + & 0.5\,x_2 & - & 0.5\,x_3 .
\end{array}
$$

Simplex Method—Second Pivot

Here's the dictionary after the first pivot:

$$\begin{array}{rcrrrrrrr} \zeta & = & 12.5 & - & 2.5\, w_1 & - & 3.5\, x_2 & + & 0.5\, x_3 \\ \hline x_1 & = & 2.5 & - & 0.5\, w_1 & - & 1.5\, x_2 & - & 0.5\, x_3 \\ w_2 & = & 1 & + & 2\, w_1 & + & 5\, x_2 & & \\ w_3 & = & 0.5 & + & 1.5\, w_1 & + & 0.5\, x_2 & - & 0.5\, x_3 . \end{array}$$

- Now, let x_3 increase.
- Of the basic variables, w_3 hits zero first.
- So, x_1 *enters* and w_3 *leaves* the basis.
- New dictionary is...

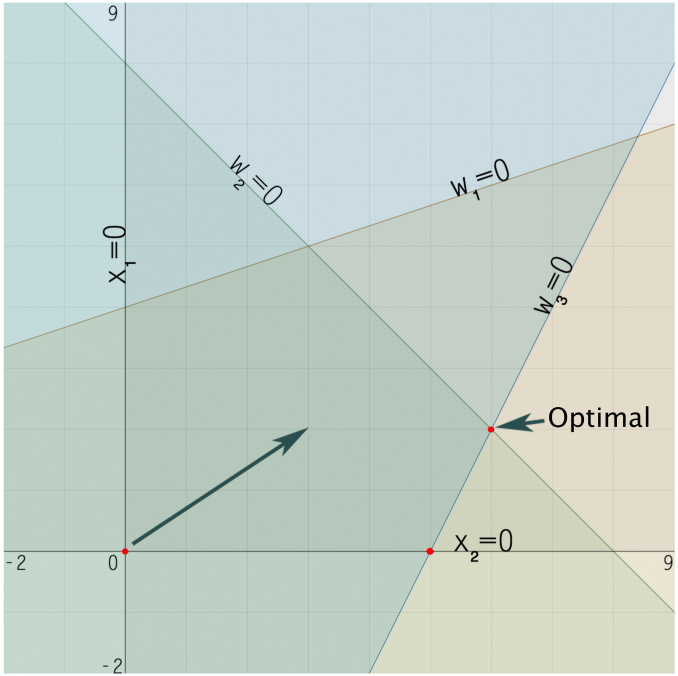
Simplex Method—Final Dictionary

$$
\begin{array}{rcl}
\zeta & = & 13 - 1\,w_1 - 3\,x_2 - 1\,w_3 \\
\hline
x_1 & = & 2 - 2\,w_1 - 2\,x_2 + w_3 \\
w_2 & = & 1 + 2\,w_1 + 5\,x_2 \\
x_3 & = & 1 + 3\,w_1 + x_2 - 2\,w_3
\end{array}
$$

- It's optimal (no pink)!
- Click here to practice the simplex method.
- Click here to solve some "challenge" problems.

Geometry

$$
\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & + & 2x_2 & & \\
\text{subject to} & -x_1 & + & 3x_2 & \leq & 12 \\
 & x_1 & + & x_2 & \leq & 8 \\
 & 2x_1 & - & x_2 & \leq & 10 \\
 & & & x_1,\ x_2 & \geq & 0.
\end{array}
$$



Unboundedness

Consider the following dictionary:

$$
\begin{array}{rcl}
\zeta &=& 5 + 1\,x_3 - 1\,x_1 \\
\hline
x_2 &=& 5 + 2\,x_3 - 3\,x_1 \\
x_4 &=& 7 - 4\,x_1 \\
x_5 &=& x_1
\end{array}
$$

- Could increase x_3 to increase obj.
- Which basic variable decreases to zero first?
- Answer: none of them, x_3 can go off to infinity, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.

Degeneracy. Solve This...

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & \\ \text{subject to} & x_1 & + & 2x_2 & \leq & 8 \\ & x_1 & - & x_2 & \leq & 4 \\ & -x_1 & + & x_2 & \leq & 4 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

Solution

maximize ζ	=	0	+	2	x_1	+	3	x_2
w_1	=	8	-	1	x_1	-	2	x_2
w_2	=	4	-	1	x_1	-	-1	x_2
w_3	=	4	-	-1	x_1	-	1	x_2

⇓ Enter: x_2, Leave: w_3

maximize ζ	=	12	+	5	x_1	+	-3	w_3
w_1	=	0	-	3	x_1	-	-2	w_3
w_2	=	8	-	0	x_1	-	1	w_3
x_2	=	4	-	-1	x_1	-	1	w_3

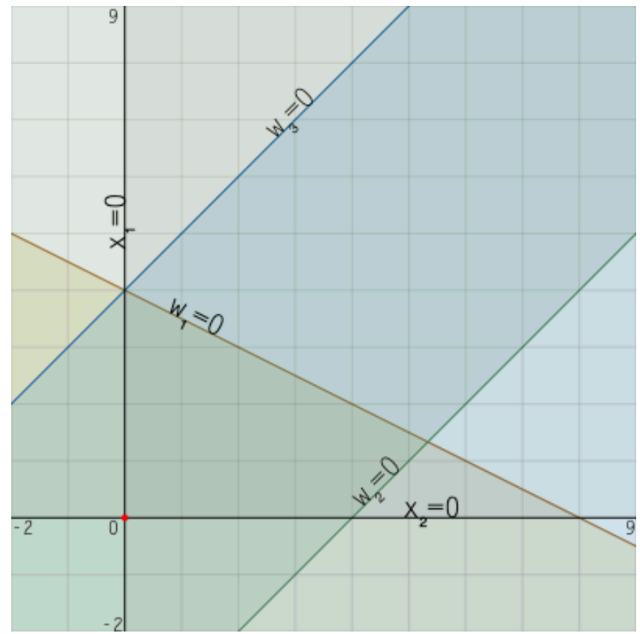
⇓ Enter: x_1, Leave: w_1

maximize ζ	=	12	+	-5/3	w_1	+	1/3	w_3
x_1	=	0	-	1/3	w_1	-	-2/3	w_3
w_2	=	8	-	0	w_1	-	1	w_3
x_2	=	4	-	1/3	w_1	-	1/3	w_3

⇓ Enter: w_3, Leave: w_2

maximize ζ	=	44/3	+	-5/3	w_1	+	-1/3	w_2
x_1	=	16/3	-	1/3	w_1	-	2/3	w_2
w_3	=	8	-	0	w_1	-	1	w_2
x_2	=	4/3	-	1/3	w_1	-	-1/3	w_2

Note: The horizontal axis, which one might call the x_1-axis, is where $x_2 = 0$ and is labeled as such.



In (x_1, x_2) coordinates, the pivots visit the following vertices:

$$(0,0) \Longrightarrow (0,1) \Longrightarrow (0,1) \Longrightarrow (4/3, 1/3).$$

Note that the second pivot went nowhere.

Degeneracy

Definitions.

A *dictionary is degenerate* if one or more “rhs”-value vanishes.

Example:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 6 & + & w_3 & + & 5x_2 & + & 4w_1 \\
\hline
x_3 & = & 1 & - & 2w_3 & - & 2x_2 & + & 3w_1 \\
w_2 & = & 4 & + & w_3 & + & x_2 & - & 3w_1 \\
x_1 & = & 3 & - & 2w_3 & & & & \\
w_4 & = & 2 & + & w_3 & & & - & w_1 \\
w_5 & = & 0 & & & - & x_2 & + & w_1
\end{array}
$$

A *pivot is degenerate* if the objective function value does not change.

Examples (based on above dictionary):

1. If x_2 enters, then w_5 must leave, pivot is degenerate.
2. If w_1 enters, then w_2 must leave, pivot is *not* degenerate.

Cycling

A *cycle* is a sequence of pivots that returns to the dictionary from which the cycle began.
Note: Every pivot in a cycle must be degenerate. Why?

Pivot Rules

A *pivot rule* is an explicit statement for how one chooses entering and leaving variables (when a choice exists).

Some Examples:

Largest-Coefficient Rule. (most common pivot rule for entering variable)

Choose the variable with the largest coefficient in the objective function.

Random Positive-Coefficient Rule.

Among all nonbasic variables having a positive coefficient, choose one at random.

First Encountered Rule.

In scanning the nonbasic variables, stop with the first one whose coefficient is positive.

Hope

Some pivot rule, such as the largest coefficient rule, will be proven never to cycle.

Hope Fades

An example that cycles using the following pivot rules:

- entering variable: largest-coefficient rule.
- leaving variable: smallest-index rule.

$$
\begin{array}{rcrcrcrcrcr}
\zeta & = & & & x_1 & - & 2x_2 & & & - & 2x_4 \\
\hline
w_1 & = & 0 & - & 0.5x_1 & + & 3.5x_2 & + & 2x_3 & - & 4x_4 \\
w_2 & = & 0 & - & 0.5x_1 & + & x_2 & + & 0.5x_3 & - & 0.5x_4
\end{array}
$$

Here's a demo of cycling...

ζ	=	0	+	1	x_1	+	-2	x_2	+	0	x_3	+	-2	x_4		
w_1	=	0	-	1/2	x_1	-	-7/2	x_2	-	-2	x_3	-	4	x_4		
w_2	=	0	-	1/2	x_1	-	-1	x_2	-	-1/2	x_3	-	1/2	x_4		

$\Downarrow$ Enter: x_1, Leave: w_1

ζ	=	0	+	-2	w_1	+	5	x_2	+	4	x_3	+	-10	x_4
x_1	=	0	-	2	w_1	-	-7	x_2	-	-4	x_3	-	8	x_4
w_2	=	0	-	-1	w_1	-	5/2	x_2	-	3/2	x_3	-	-7/2	x_4

$\Downarrow$ Enter: x_2, Leave: w_2

ζ	=	0	+	0	w_1	+	-2	w_2	+	1	x_3	+	-3	x_4
x_1	=	0	-	-4/5	w_1	-	14/5	w_2	-	1/5	x_3	-	-9/5	x_4
x_2	=	0	-	-2/5	w_1	-	2/5	w_2	-	3/5	x_3	-	-7/5	x_4

$\Downarrow$ Enter: x_3, Leave: x_1

ζ	=	0	+	4	w_1	+	-16	w_2	+	-5	x_1	+	6	x_4
x_3	=	0	-	-4	w_1	-	14	w_2	-	5	x_1	-	-9	x_4
x_2	=	0	-	2	w_1	-	-8	w_2	-	-3	x_1	-	4	x_4

$\Downarrow$ Enter: x_4, Leave: x_2

$$
\begin{array}{rcccccccccccccc}
\zeta & = & 0 & + & 1 & w_1 & + & -4 & w_2 & + & -1/2 & x_1 & + & -3/2 & x_2 \\
x_3 & = & 0 & - & 1/2 & w_1 & - & -4 & w_2 & - & -7/4 & x_1 & - & 9/4 & x_2 \\
x_4 & = & 0 & - & 1/2 & w_1 & - & -2 & w_2 & - & -3/4 & x_1 & - & 1/4 & x_2
\end{array}
$$

$\Downarrow$ Enter: w_1, Leave: x_3

$$
\begin{array}{rcccccccccccccc}
\zeta & = & 0 & + & -2 & x_3 & + & 4 & w_2 & + & 3 & x_1 & + & -6 & x_2 \\
w_1 & = & 0 & - & 2 & x_3 & - & -8 & w_2 & - & -7/2 & x_1 & - & 9/2 & x_2 \\
x_4 & = & 0 & - & -1 & x_3 & - & 2 & w_2 & - & 1 & x_1 & - & -2 & x_2
\end{array}
$$

$\Downarrow$ Enter: w_2, Leave: x_4

$$
\begin{array}{rcccccccccccccc}
\zeta & = & 0 & + & 0 & x_3 & + & -2 & x_4 & + & 1 & x_1 & + & -2 & x_2 \\
w_1 & = & 0 & - & -2 & x_3 & - & 4 & x_4 & - & 1/2 & x_1 & - & -7/2 & x_2 \\
w_2 & = & 0 & - & -1/2 & x_3 & - & 1/2 & x_4 & - & 1/2 & x_1 & - & -1 & x_2
\end{array}
$$

If the simplex method fails to terminate, then it must cycle.

Perturbation Method

Whenever a vanishing "rhs" appears perturb it.
If there are lots of them, say k, perturb them all.
Make the perturbations at different *scales*:

$$\text{other data} \gg \epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_k > 0.$$

An Example.

$$
\begin{array}{llllllllllllll}
\text{maximize} & \zeta & = & & 0\,\epsilon_1 & + & 0\,\epsilon_2 & + & 0\,\epsilon_3 & + & 2\,x_1 & + & 3\,x_2 \\
\text{subject to:} & w_1 & = & 0 & +\ 1\,\epsilon_1 & & & & & - & -1\,x_1 & - & 1\,x_2 \\
 & w_2 & = & 0 & & + & 1\,\epsilon_2 & & & - & -3\,x_1 & - & 1\,x_2 \\
 & w_3 & = & 0 & & & & + & 1\,\epsilon_3 & - & 4\,x_1 & - & -1\,x_2
\end{array}
$$

Entering variable: x_2
Leaving variable: w_2

$$
\begin{array}{llll}
\text{maximize} & \zeta & = & 0\,\epsilon_1 + 3\,\epsilon_2 + 0\,\epsilon_3 + 11\,x_1 + -3\,w_2 \\
\text{subject to:} & w_1 & = & 0 + 1\,\epsilon_1 + -1\,\epsilon_2 - 2\,x_1 - -1\,w_2 \\
 & x_2 & = & 0 + 1\,\epsilon_2 - -3\,x_1 - 1\,w_2 \\
 & w_3 & = & 0 + 1\,\epsilon_2 + 1\,\epsilon_3 - 1\,x_1 - 1\,w_2
\end{array}
$$

Entering variable: x_1
Leaving variable: w_3

$$
\begin{array}{llll}
\text{maximize} & \zeta & = & 0\,\epsilon_1 + 14\,\epsilon_2 + 11\,\epsilon_3 + -11\,w_3 + -14\,w_2 \\
\text{subject to:} & w_1 & = & 0 + 1\,\epsilon_1 + -3\,\epsilon_2 + -2\,\epsilon_3 - -2\,w_3 - -3\,w_2 \\
 & x_2 & = & 0 + 4\,\epsilon_2 + 3\,\epsilon_3 - 3\,w_3 - 4\,w_2 \\
 & x_1 & = & 0 + 1\,\epsilon_2 + 1\,\epsilon_3 - 1\,w_3 - 1\,w_2
\end{array}
$$

DONE!

Perturbation Method Applied to Cycling Example

ζ	=	0	+	0	ϵ_1 +	0	ϵ_2 +	1	x_1	+	−2	x_2	+	0	x_3	+	−2	x_4
w_1	=	0	+	1	ϵ_1 +	0	ϵ_2 −	1/2	x_1	−	−1	x_2	−	−1/2	x_3	−	1/2	x_4
w_2	=	0	+	0	ϵ_1 +	1	ϵ_2 −	1/2	x_1	−	−7/2	x_2	−	−2	x_3	−	4	x_4

$\Downarrow$ x_1 enters, w_2 leaves

ζ	=	0	+	0	ϵ_1 +	2	ϵ_2 +	−2	w_2	+	5	x_2	+	4	x_3	+	−10	x_4
w_1	=	0	+	1	ϵ_1 +	−1	ϵ_2 −	−1	w_2	−	5/2	x_2	−	3/2	x_3	−	−7/2	x_4
x_1	=	0	+	0	ϵ_1 +	2	ϵ_2 −	2	w_2	−	−7	x_2	−	−4	x_3	−	8	x_4

$\Downarrow$ x_2 enters, w_1 leaves

ζ	=	0	+	2	ϵ_1 +	0	ϵ_2 +	0	w_2	+	−2	w_1	+	1	x_3	+	−3	x_4
x_2	=	0	+	2/5	ϵ_1 +	−2/5	ϵ_2 −	−2/5	w_2	−	2/5	w_1	−	3/5	x_3	−	−7/5	x_4
x_1	=	0	+	14/5	ϵ_1 +	−4/5	ϵ_2 −	−4/5	w_2	−	14/5	w_1	−	1/5	x_3	−	−9/5	x_4

$\Downarrow$ x_3 enters, x_2 leaves

ζ	=	0	+	8/3	ϵ_1 +	−2/3	ϵ_2 +	2/3	w_2	+	−8/3	w_1	+	−5/3	x_2	+	−2/3	x_4
x_3	=	0	+	2/3	ϵ_1 +	−2/3	ϵ_2 −	−2/3	w_2	−	2/3	w_1	−	5/3	x_2	−	−7/3	x_4
x_1	=	0	+	8/3	ϵ_1 +	−2/3	ϵ_2 −	−2/3	w_2	−	8/3	w_1	−	−1/3	x_2	−	−4/3	x_4

$\Downarrow$ w_2 enters, problem unbounded!

Note: objective function increases with every pivot: $0 < 2\varepsilon_2 < 2\varepsilon_1 < \frac{8}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2$

Efficiency of the Simplex Method

Question:

Given a problem of a certain size, how long will it take to solve it?

Two Kinds of Answers:

- *Average Case.* How long for a *typical* problem.
- *Worst Case.* How long for the *hardest* problem.

Average Case.

- Mathematically difficult (define average!).
- Empirical studies.

Worst Case.

- Mathematically tractible (sometimes).
- Limited value.

Worst case example...

Klee–Minty Problem (1972)

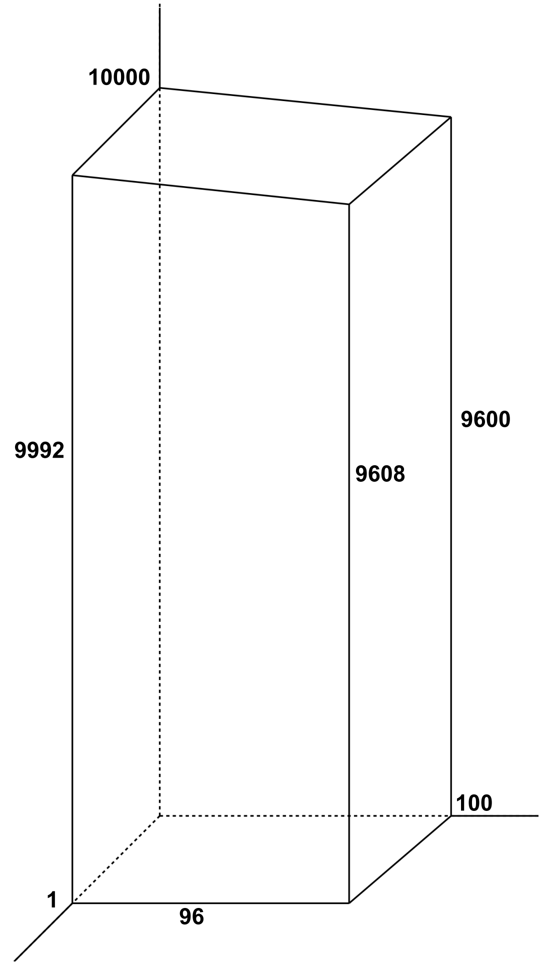
$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} 2^{n-j} x_j & \\ \text{subject to} & \displaystyle 2\sum_{j=1}^{i-1} 2^{i-j} x_j + x_i \le 100^{i-1} & i = 1, 2, \ldots, n \\ & x_j \ge 0 & j = 1, 2, \ldots, n. \end{array}$$

Example $n = 3$:

$$\begin{array}{lrcrcrcr} \text{maximize} & 4x_1 & + & 2x_2 & + & x_3 & & \\ \text{subj. to} & x_1 & & & & & \le & 1 \\ & 4x_1 & + & x_2 & & & \le & 100 \\ & 8x_1 & + & 4x_2 & + & x_3 & \le & 10000 \\ & & & x_1, & x_2, & x_3 & \ge & 0. \end{array}$$

A Distorted Cube

Case $n = 3$:



Constraints represent a "minor" distortion to an n-dimensional hypercube:

$$\begin{aligned} 0 \leq \; & x_1 \leq 1 \\ 0 \leq \; & x_2 \leq 100 \\ & \vdots \\ 0 \leq \; & x_n \leq 100^{n-1}. \end{aligned}$$

Largest coefficient rule: 2^{n-1} iterations.

Smallest coefficient rule: 1 iteration.

Complexity

n	n^2	n^3	2^n
1	1	1	2
2	4	8	4
3	9	27	8
4	16	64	16
5	25	125	32
6	36	216	64
7	49	343	128
8	64	512	256
9	81	729	512
10	100	1000	1024
12	144	1728	4096
14	196	2744	16384
16	256	4096	65536
18	324	5832	262144
20	400	8000	1048576
22	484	10648	4194304
24	576	13824	16777216
26	676	17576	67108864
28	784	21952	268435456
30	900	27000	1073741824

Sorting: fast algorithm $= n \log n$,
slow algorithm $= n^2$
Matrix times vector: n^2
Matrix times matrix: n^3
Matrix inversion: n^3
Simplex Method:

- Worst case: $n^2 2^n$ operations.
- Average case: n^3 operations.
- Open question:

  Does there exist a variant of the simplex method whose worst case performance is polynomial?

Linear Programming:

- *Theorem:* There exists an algorithm whose worst case performance is $n^{3.5}$ operations.

Duality

Every Problem:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\
& x_j \geq 0 & j = 1, 2, \ldots, n,
\end{array}
$$

Has a Dual:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\
\text{subject to} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\
& y_i \geq 0 & i = 1, 2, \ldots, m.
\end{array}
$$

Dual of Dual

Primal Problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{array}$$

Original problem is called the *primal problem*.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual in "Standard" Form:

$$\begin{array}{lll} -\text{maximize} & \displaystyle\sum_{i=1}^{m} -b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} -a_{ij} y_i \leq -c_j & j = 1, \ldots, n \\ & y_i \geq 0 & i = 1, \ldots, m \end{array}$$

Dual is "negative transpose" of primal.

Theorem Dual of dual is primal.

Weak Duality Theorem

If $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual, then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

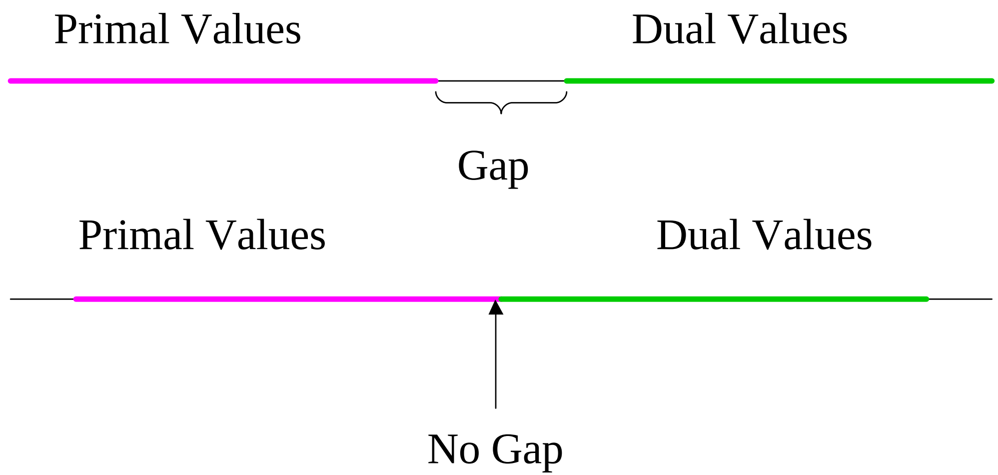
Proof.

$$\begin{aligned}
\sum_j c_j x_j &\leq \sum_j \left(\sum_i y_i a_{ij} \right) x_j \\
&= \sum_{i,j} y_i a_{ij} x_j \\
&= \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\
&\leq \sum_i b_i y_i.
\end{aligned}$$

Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?



Answer is provided by the Strong Duality Theorem (coming later).

Simplex Method and Duality

A Primal Problem:

$$\begin{array}{rcrrrrrrrrr} \zeta & = & & + & 4 & x_1 & + & 1 & x_2 & + & 3 & x_3 \\ \hline w_1 & = & 1 & - & & x_1 & - & 4 & x_2 & & & \\ w_2 & = & 3 & - & 3 & x_1 & + & & x_2 & - & & x_3 \end{array}$$

Its Dual:

$$\begin{array}{rcrrrrrrr} -\xi & = & & - & 1 & y_1 & - & 3 & y_2 \\ \hline z_1 & = & -4 & + & & y_1 & + & 3 & y_2 \\ z_2 & = & -1 & + & 4 & y_1 & - & & y_2 \\ z_3 & = & -3 & & & & + & & y_2 \end{array}$$

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: x_3 enters, w_2 leaves.

Make analogous pivot in dual: z_3 leaves, y_2 enters.

After First Pivot

Primal (feasible):

$$\begin{array}{rcrcrcrcr} \zeta & = & 9 & - & 5\,x_1 & + & 4\,x_2 & - & 3\,w_2 \\ \hline w_1 & = & 1 & - & x_1 & - & 4\,x_2 & + & \\ x_3 & = & 3 & - & 3\,x_1 & + & x_2 & - & w_2 \end{array}$$

Dual (still not feasible):

$$\begin{array}{rcrcrcr} -\xi & = & -9 & - & 1\,y_1 & - & 3\,z_3 \\ \hline z_1 & = & 5 & + & y_1 & + & 3\,z_3 \\ z_2 & = & -4 & + & 4\,y_1 & - & z_3 \\ y_2 & = & 3 & + & & + & z_3 \end{array}$$

Note: *negative transpose property intact.*

Again, use primal to pick pivot: x_2 enters, w_1 leaves.

Make analogous pivot in dual: z_2 leaves, y_1 enters.

After Second Pivot

Primal (Is Optimal):

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 10 & - & 6 & x_1 & - & 1 & w_1 & - & 3 & w_2 \\
\hline
x_2 & = & 0.25 & - & 0.25 & x_1 & - & 0.25 & w_1 & & & \\
x_3 & = & 3.25 & - & 3.25 & x_1 & - & 0.25 & w_1 & - & & w_2
\end{array}
$$

Dual (Is Optimal):

$$
\begin{array}{rcrcrcrcr}
-\xi & = & -10 & - & 0.25 & z_2 & - & 3.25 & z_3 \\
\hline
z_1 & = & 6 & + & 0.25 & z_2 & + & 3.25 & z_3 \\
y_1 & = & 1 & + & 0.25 & z_2 & + & 0.25 & z_3 \\
y_2 & = & 3 & & & & + & 1 & z_3
\end{array}
$$

Note: *negative transpose property intact.*

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.

Algebra of a Pivot

A primal pivot:

d	c	
b	a	

$\xrightarrow{\text{pivot}}$

$d - \frac{bc}{a}$	c/a	
$-b/a$	$1/a$	

The corresponding dual pivot:

$-d$	$-b$	
$-c$	$-a$	

$\xrightarrow{\text{pivot}}$

$-d + \frac{bc}{a}$	b/a	
$-c/a$	$-1/a$	

Strong Duality Theorem

Conclusion on previous slide is the essence of the *strong duality theorem* which we now state:

Theorem. *If the primal problem has an optimal solution,*

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Paraphrase:

If primal has an optimal solution, then there is no duality gap.

Duality Gap

Four possibilities:

- Primal optimal $\Longrightarrow$ dual optimal (no gap) (Strong Duality Theorem)
- Primal unbounded $\Longrightarrow$ dual infeasible (no gap) (Weak Duality Theorem)
- Primal infeasible $\Longleftarrow$ dual unbounded (no gap) (Weak Duality Theorem)
- Primal infeasible dual infeasible (infinite gap) See example below

Example of *infinite gap*:

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & x_1 & - & x_2 & \leq & 1 \\
& -x_1 & + & x_2 & \leq & -2 \\
& & & x_1, \; x_2 & \geq & 0.
\end{array}
$$

The Dual Simplex Method

We begin with an example:

$$\begin{array}{lrcrcr} \text{maximize} & -x_1 & - & x_2 & & \\ \text{subject to} & -2x_1 & - & x_2 & \leq & 4 \\ & -2x_1 & + & 4x_2 & \leq & -8 \\ & -x_1 & + & 3x_2 & \leq & -7 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

The dual of this problem is

$$\begin{array}{lrcrcrcr} \text{minimize} & 4y_1 & - & 8y_2 & - & 7y_3 & & \\ \text{subject to} & -2y_1 & - & 2y_2 & - & y_3 & \geq & -1 \\ & -y_1 & + & 4y_2 & + & 3y_3 & \geq & -1 \\ & & & & y_1,\ y_2, & y_3 & \geq & 0. \end{array}$$

Using "dictionary" notation...

(P)
$$\begin{array}{rcrrrrrr} \zeta & = & & - & 1\,x_1 & - & 1\,x_2 \\ \hline w_1 & = & 4 & + & 2\,x_1 & + & x_2 \\ w_2 & = & -8 & + & 2\,x_1 & - & 4\,x_2 \\ w_3 & = & -7 & + & x_1 & - & 3\,x_2 \end{array}$$

(D)
$$\begin{array}{rcrrrrrrr} -\xi & = & & - & 4\,y_1 & + & 8\,y_2 & + & 7\,y_3 \\ \hline z_1 & = & 1 & - & 2\,y_1 & - & 2\,y_2 & - & y_3 \\ z_2 & = & 1 & - & y_1 & + & 4\,y_2 & + & 3\,y_3 \end{array}$$

The Dual Simplex Method

Apply the simplex method to the dual problem and make the analogous pivots on the primal...

(P)
$$
\begin{array}{rcl}
\zeta & = & -1\,x_1 - 1\,x_2 \\
\hline
w_1 & = & 4 + 2\,x_1 + x_2 \\
w_2 & = & -8 + 2\,x_1 - 4\,x_2 \\
w_3 & = & -7 + x_1 - 3\,x_2
\end{array}
$$

(D)
$$
\begin{array}{rcl}
-\xi & = & -4\,y_1 + 8\,y_2 + 7\,y_3 \\
\hline
z_1 & = & 1 - 2\,y_1 - 2\,y_2 - y_3 \\
z_2 & = & 1 - y_1 + 4\,y_2 + 3\,y_3
\end{array}
$$

(P)
$$
\begin{array}{rcl}
\zeta & = & -4 - 0.5\,w_2 - 3\,x_2 \\
\hline
w_1 & = & 12 + w_2 + 5\,x_2 \\
x_1 & = & 4 + 0.5\,w_2 + 2\,x_2 \\
w_3 & = & -3 + 0.5\,w_2 - x_2
\end{array}
$$

(D)
$$
\begin{array}{rcl}
-\xi & = & 4 - 12\,y_1 - 4\,z_1 + 3\,y_3 \\
\hline
y_2 & = & 0.5 - 1\,y_1 - 0.5\,z_1 - 0.5\,y_3 \\
z_2 & = & 3 - 5\,y_1 - 2\,z_1 + 1\,y_3
\end{array}
$$

(P)
$$
\begin{array}{rcl}
\zeta & = & -7 - 1\,w_3 - 4\,x_2 \\
\hline
w_1 & = & 18 + 2\,w_3 + 7\,x_2 \\
x_1 & = & 7 + w_3 + 3\,x_2 \\
w_2 & = & 6 + 2\,w_3 + 2\,x_2
\end{array}
$$

(D)
$$
\begin{array}{rcl}
-\xi & = & 7 - 18\,y_1 - 7\,z_1 - 6\,y_2 \\
\hline
y_3 & = & 1 - 2\,y_1 - z_1 - 2\,y_2 \\
z_2 & = & 4 - 7\,y_1 - 3\,z_1 - 2\,y_2
\end{array}
$$

Both primal and dual are optimal!

Two Phase Method

Dual-Based Phase 1 Method

Example:

Pivot Tool -- Advanced Version

$$
\begin{array}{lcrcrlcrlcrl}
\text{maximize } \zeta & = & 0 & + & 3 & x_1 & + & 6 & x_2 & + & -6 & x_3 \\
\zeta_0 & = & & + & -1 & x_1 & + & -1 & x_2 & + & -1 & x_3 \\
w_1 & = & -2 & - & 1 & x_1 & - & -1 & x_2 & - & -2 & x_3 \\
w_2 & = & -1 & - & 0 & x_1 & - & 1 & x_2 & - & -2 & x_3 \\
w_3 & = & 6 & - & 1 & x_1 & - & 4 & x_2 & - & 0 & x_3 \\
w_4 & = & 3 & - & -2 & x_1 & - & -1 & x_2 & - & 5 & x_3
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$$

Seed = 4

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it's *dual feasible*.

Phase I—First Pivot: w_1 leaves, x_3 enters.

Let's go pivoting...

Recall initial dictionary:

Pivot Tool -- Advanced Version

maximize ζ	=	0	+	3	x_1	+	6	x_2	+	-6	x_3
ζ_0	=		+	-1	x_1	+	-1	x_2	+	-1	x_3
w_1	=	-2	-	1	x_1	-	-1	x_2	-	-2	x_3
w_2	=	-1	-	0	x_1	-	1	x_2	-	-2	x_3
w_3	=	6	-	1	x_1	-	4	x_2	-	0	x_3
w_4	=	3	-	-2	x_1	-	-1	x_2	-	5	x_3

$x_1, x_2, x_3, \; w_1, w_2, w_3, w_4 \geq 0$

Dual pivot: w_1 leaves, x_3 enters.

After pivot:

Pivot Tool -- Advanced Version

maximize ζ	=	-6	+	0	x_1	+	9	x_2	+	-3	w_1
ζ_0	=		+	-3/2	x_1	+	-1/2	x_2	+	-1/2	w_1
x_3	=	1	-	-1/2	x_1	-	1/2	x_2	-	-1/2	w_1
w_2	=	1	-	-1	x_1	-	2	x_2	-	-1	w_1
w_3	=	6	-	1	x_1	-	4	x_2	-	0	w_1
w_4	=	-2	-	1/2	x_1	-	-7/2	x_2	-	5/2	w_1

$x_1, x_2, x_3, \; w_1, w_2, w_3, w_4 \geq 0$

Recall current dictionary:

Pivot Tool -- Advanced Version

maximize ζ	=	-6	+	0	x_1	+	9	x_2	+	-3 w_1
ζ_0	=		+	-3/2	x_1	+	-1/2	x_2	+	-1/2 w_1
x_3	=	1	-	-1/2	x_1	-	1/2	x_2	-	-1/2 w_1
w_2	=	1	-	-1	x_1	-	2	x_2	-	-1 w_1
w_3	=	6	-	1	x_1	-	4	x_2	-	0 w_1
w_4	=	-2	-	1/2	x_1	-	-7/2	x_2	-	5/2 w_1

$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Dual pivot: w_4 leaves, x_2 enters.

After pivot:

Pivot Tool -- Advanced Version

maximize ζ	=	-6/7	+	9/7	x_1	+	18/7	w_4	+	24/7 w_1
ζ_0	=		+	-11/7	x_1	+	-1/7	w_4	+	-6/7 w_1
x_3	=	5/7	-	-3/7	x_1	-	1/7	w_4	-	-1/7 w_1
w_2	=	-1/7	-	-5/7	x_1	-	4/7	w_4	-	3/7 w_1
w_3	=	26/7	-	11/7	x_1	-	8/7	w_4	-	20/7 w_1
x_2	=	4/7	-	-1/7	x_1	-	-2/7	w_4	-	-5/7 w_1

$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Recall current dictionary:

Pivot Tool -- Advanced Version

maximize ζ	=	-6/7	+	9/7	x_1	+	18/7	w_4	+	24/7	w_1
ζ_0	=		+	-11/7	x_1	+	-1/7	w_4	+	-6/7	w_1
x_3	=	5/7	-	-3/7	x_1	-	1/7	w_4	-	-1/7	w_1
w_2	=	-1/7	-	-5/7	x_1	-	4/7	w_4	-	3/7	w_1
w_3	=	26/7	-	11/7	x_1	-	8/7	w_4	-	20/7	w_1
x_2	=	4/7	-	-1/7	x_1	-	-2/7	w_4	-	-5/7	w_1

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Dual pivot: w_2 leaves, x_1 enters.

After pivot:

Feasible!

Pivot Tool -- Advanced Version

maximize ζ	=	-3/5	+	9/5	w_2	+	18/5	w_4	+	21/5	w_1
ζ_0	=		+	-11/5	w_2	+	-7/5	w_4	+	-9/5	w_1
x_3	=	4/5	-	-3/5	w_2	-	-1/5	w_4	-	-2/5	w_1
x_1	=	1/5	-	-7/5	w_2	-	-4/5	w_4	-	-3/5	w_1
w_3	=	17/5	-	11/5	w_2	-	12/5	w_4	-	19/5	w_1
x_2	=	3/5	-	-1/5	w_2	-	-2/5	w_4	-	-4/5	w_1

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Current dictionary is *feasible*:

Pivot Tool -- Advanced Version

maximize	ζ	=	-3/5	+	9/5	w_2	+	18/5	w_4	+	21/5	w_1
	ζ_0	=		+	-11/5	w_2	+	-7/5	w_4	+	-9/5	w_1
	x_3	=	4/5	-	-3/5	w_2	-	-1/5	w_4	-	-2/5	w_1
	x_1	=	1/5	-	-7/5	w_2	-	-4/5	w_4	-	-3/5	w_1
	w_3	=	17/5	-	11/5	w_2	-	12/5	w_4	-	19/5	w_1
	x_2	=	3/5	-	-1/5	w_2	-	-2/5	w_4	-	-4/5	w_1

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Ignore fake objective. Use the real objective. Primal pivot: w_1 enters, w_3 leaves.

After pivot:

Pivot Tool -- Advanced Version

maximize	ζ	=	60/19	+	-12/19	w_2	+	18/19	w_4	+	-21/19	w_3
	ζ_0	=		+	-22/19	w_2	+	-5/19	w_4	+	9/19	w_3
	x_3	=	22/19	-	-7/19	w_2	-	1/19	w_4	-	2/19	w_3
	x_1	=	14/19	-	-20/19	w_2	-	-8/19	w_4	-	3/19	w_3
	w_1	=	17/19	-	11/19	w_2	-	12/19	w_4	-	5/19	w_3
	x_2	=	25/19	-	5/19	w_2	-	2/19	w_4	-	4/19	w_3

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Getting close:

Pivot Tool -- Advanced Version

maximize	ζ	=	60/19	+	-12/19	w_2	+	18/19	w_4	+	-21/19	w_3
	ζ_0	=		+	-22/19	w_2	+	-5/19	w_4	+	9/19	w_3
	x_3	=	22/19	-	-7/19	w_2	-	1/19	w_4	-	2/19	w_3
	x_1	=	14/19	-	-20/19	w_2	-	-8/19	w_4	-	3/19	w_3
	w_1	=	17/19	-	11/19	w_2	-	12/19	w_4	-	5/19	w_3
	x_2	=	25/19	-	5/19	w_2	-	2/19	w_4	-	4/19	w_3

$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Primal pivot: w_4 enters, w_1 leaves.

After pivot:

Optimal!

Pivot Tool -- Advanced Version

maximize	ζ	=	9/2	+	-3/2	w_2	+	-3/2	w_1	+	-3/2	w_3
	ζ_0	=		+	-11/12	w_2	+	5/12	w_1	+	7/12	w_3
	x_3	=	13/12	-	-5/12	w_2	-	-1/12	w_1	-	1/12	w_3
	x_1	=	4/3	-	-2/3	w_2	-	2/3	w_1	-	1/3	w_3
	w_4	=	17/12	-	11/12	w_2	-	19/12	w_1	-	5/12	w_3
	x_2	=	7/6	-	1/6	w_2	-	-1/6	w_1	-	1/6	w_3

$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Parametric Self-Dual Method

An Example:

$$\begin{array}{lrcrcr} \text{maximize} & -2x_1 & + & 3x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & -1 \\ & -x_1 & - & 2x_2 & \leq & -2 \\ & & & x_2 & \leq & 1 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

Initial Dictionary:

$$\begin{array}{rcrcrcr} \zeta & = & & - & 2x_1 & + & 3x_2 \\ \hline w_1 & = & -1 & + & x_1 & - & x_2 \\ w_2 & = & -2 & + & x_1 & + & 2x_2 \\ w_3 & = & 1 & & & - & x_2 \end{array}$$

Note: neither primal nor dual feasible.

Perturb

Introduce a parameter μ and perturb:

$$\begin{array}{rcrrrcrr} \zeta & = & & (-2-\mu) & x_1 & + & (3-\mu) & x_2 \\ \hline w_1 & = & -1+\mu & + & x_1 & - & & x_2 \\ w_2 & = & -2+\mu & + & x_1 & + & 2 & x_2 \\ w_3 & = & 1+\mu & & & - & & x_2 \end{array}$$

For μ *large*, dictionary is *optimal*.
Question: For which μ values is dictionary optimal?
Answer:

$$\begin{array}{rcrcrl} 2 & + & \mu & \geq & 0 & \\ -3 & + & \mu & \geq & 0 & * \\ \hline -1 & + & \mu & \geq & 0 & * \\ -2 & + & \mu & \geq & 0 & * \\ 1 & + & \mu & \geq & 0 & \end{array}$$

Note: only those marked with (*) give inequalities that omit $\mu = 0$.
Tightest:

$$\mu \geq 3$$

Achieved by: objective function's perturbation on x_2.
Let x_2 *enter*.

Who Leaves?

Do ratio test using current lowest μ value, i.e. $\mu = 3$:

$$\begin{array}{rcccrcc} -1 & + & 3 & - & x_2 & \geq & 0 \\ -2 & + & 3 & + & 2x_2 & \geq & 0 \\ 1 & + & 3 & - & x_2 & \geq & 0 \end{array}$$

Tightest:

$$-1 + 3 - x_2 \geq 0.$$

Achieved by: constraint containing basic variable w_1.

Let w_1 leave.

After the Pivot

$$
\begin{array}{rcrrl}
\zeta & = & -3 + 4\mu - \mu^2 & + (1 - 2\mu)\, x_1 & + (-3 + \mu)\, w_1 \\
\hline
x_2 & = & -1 + \mu & + \quad x_1 & - \quad w_1 \\
w_2 & = & -4 + 3\mu & + \quad 3\, x_1 & - \quad 2\, w_1 \\
w_3 & = & 2 & - \quad x_1 & + \quad w_1
\end{array}
$$

This dictionary is optimal for

$$4/3 \leq \mu \leq 3$$

The lower bound comes from the second constraint.

To continue, we'll do a dual pivot using w_2 as the *leaving variable*.

With this choice for the leaving variable, the *entering variable* is x_1.

After the Second Pivot

$$
\begin{array}{rcl}
\zeta & = & -\frac{5}{3}+\frac{1}{3}\mu+\mu^2 + \left(\frac{1}{3} - \frac{2}{3}\mu\right) w_2 + \left(-\frac{7}{3} - \frac{1}{3}\mu\right) w_1 \\
\hline
x_2 & = & \frac{1}{3} + \frac{1}{3} w_2 - \frac{1}{3} w_1 \\
x_1 & = & \frac{4}{3} - \mu + \frac{1}{3} w_2 + \frac{2}{3} w_1 \\
w_3 & = & \frac{2}{3} + \mu - \frac{1}{3} w_2 + \frac{1}{3} w_1
\end{array}
$$

This dictionary is optimal for

$$1/2 \leq \mu \leq 4/3$$

We've got to do (at least) one more pivot.

The lower bound comes from the objective coefficient on x_1.

To continue, we'll do a primal pivot using w_2 as the *entering variable*.

With this choice for the leaving variable, the *leaving variable* is w_3.

After the Third Pivot

$$
\begin{array}{rcllr}
\zeta & = & -1 & -\mu^2 + (-1+2\mu)\, w_3 & + (-2-\mu)\, w_1 \\
\hline
x_2 & = & 1+\mu & - \quad\quad w_3 & \\
x_1 & = & 2 & - \quad\quad w_3 & + \quad w_1 \\
w_2 & = & 2+3\mu & - \quad 3\, w_3 & + \quad w_1
\end{array}
$$

This dictionary is optimal for

$$-2/3 \leq \mu \leq 1/2$$

This dictionary is optimal for $\mu = 0$. Yay!

Setting $\mu = 0$, we see that the answer to the problem is at

$$(x_1, x_2) = (2, 1).$$

Here's a Screenshot from the Advanced Pivot Tool

Hint | Undo

Number format: Fraction — Zero Visibility: Dimmed

Current Dictionary

maximize
$$\zeta = -1 + 3\mu + -1\, w_3 + -2\, w_1$$
$$\quad + -3\mu + -1\mu^2 + 2\mu\, w_3 + -1\mu\, w_1$$

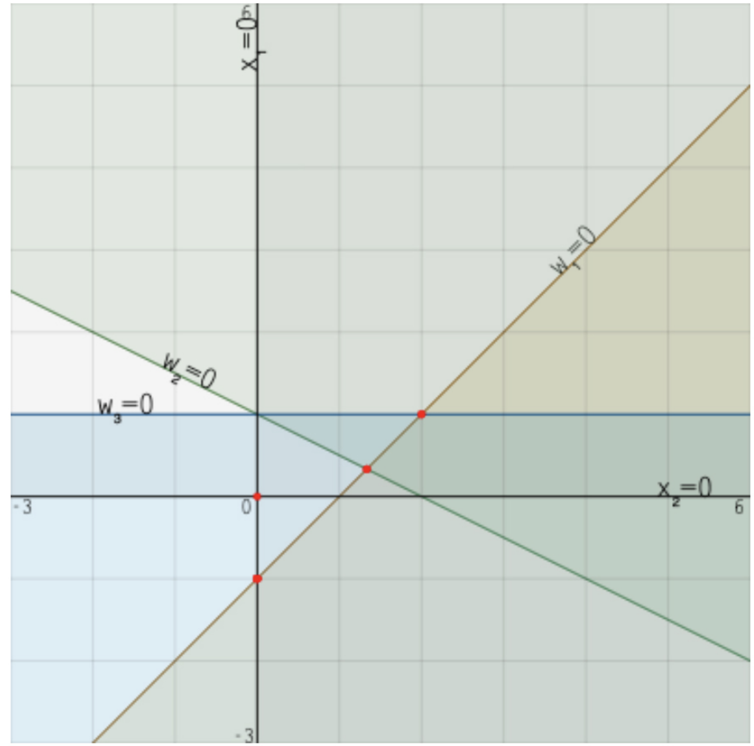
subject to:
$$x_2 = 1 + 1\mu - 1\, w_3 - 0\, w_1$$
$$x_1 = 2 + 0\mu - 1\, w_3 - -1\, w_1$$
$$w_2 = 2 + 3\mu - 3\, w_3 - -1\, w_1$$

$$x_1,\ x_2,\quad w_1,\ w_2,\ w_3 \geq 0$$

$-2/3 \leq \mu \leq 1/2$

Optimal | Primal Infeas | Dual Infeas

Pick a judge: Bart Simpson



Top Ten Reasons to Like this Method

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Okay, there are only 6 items in the list. SORRY.

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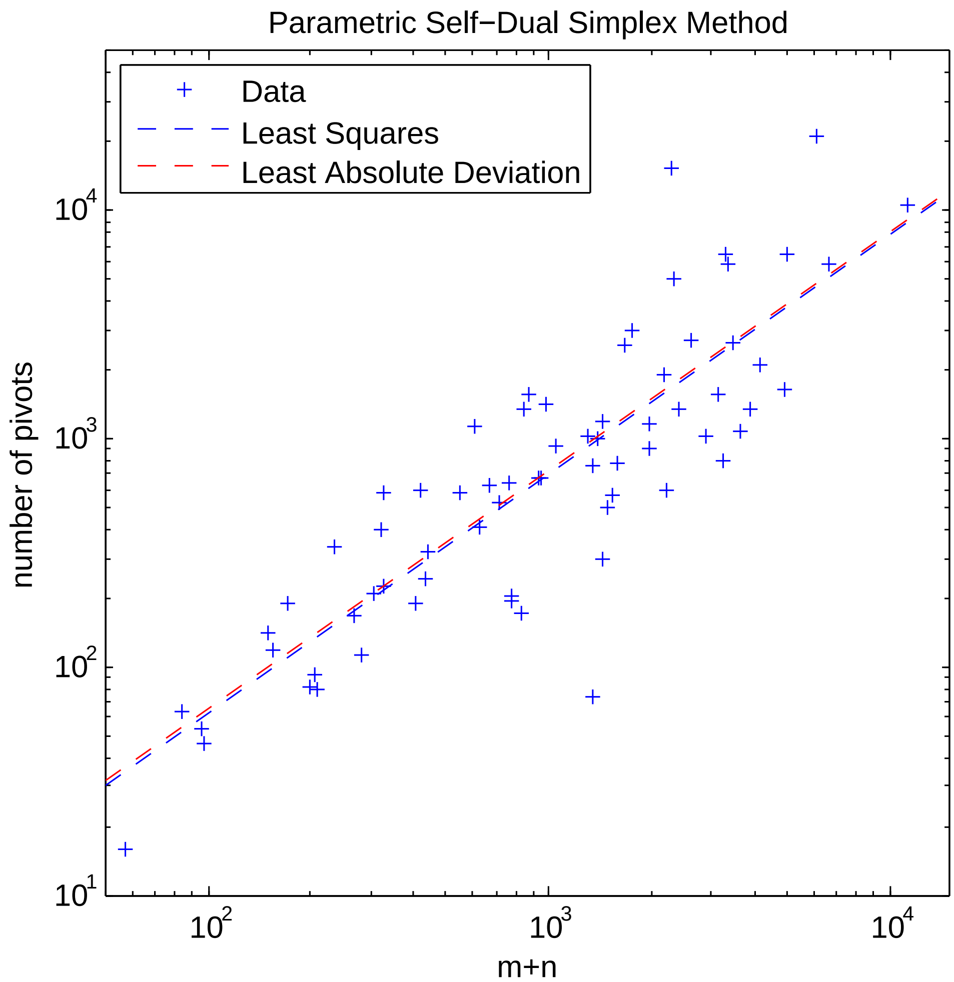
Parametric Self-Dual Simplex Method

Thought experiment:

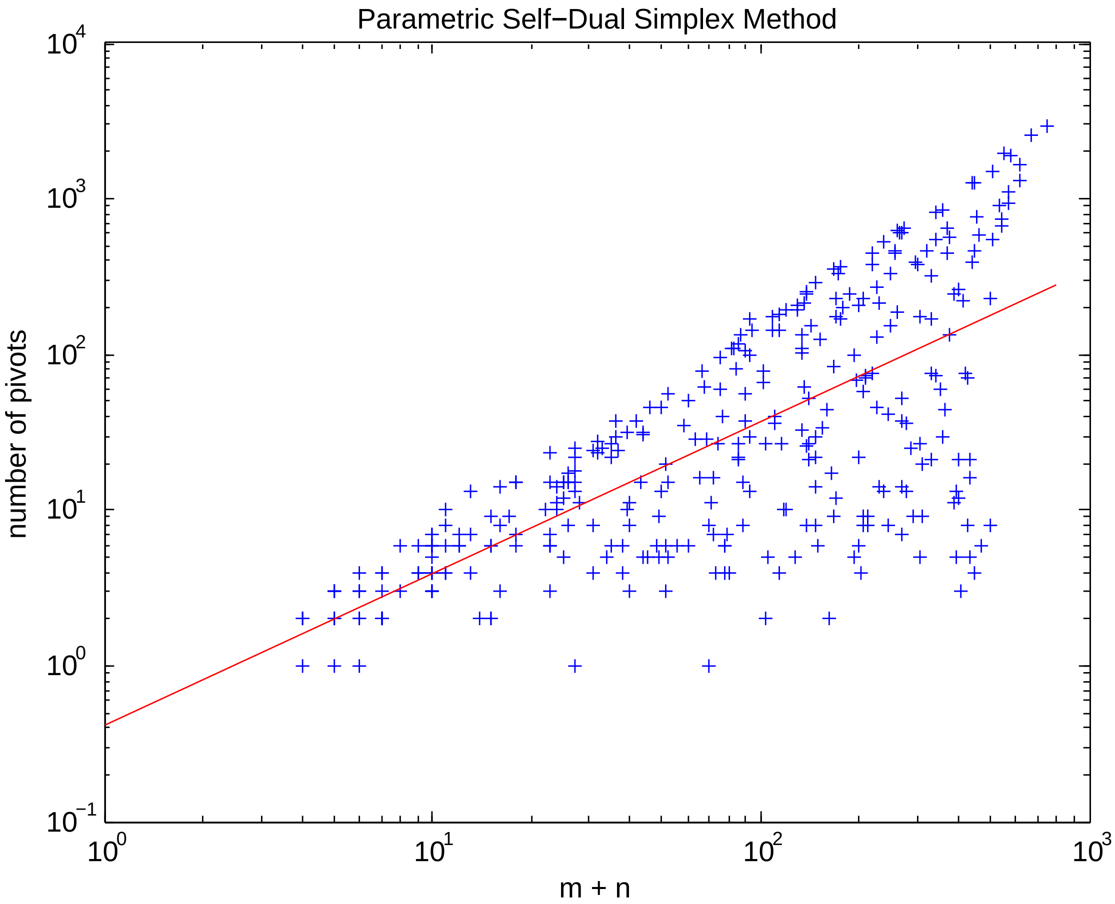
- μ starts at ∞.
- In reducing μ, there are $n + m$ barriers.
- At each iteration, one barrier is passed—the others move about "randomly".
- To get μ to zero, we must on average pass half the barriers.
- Therefore, on average the algorithm probably takes about $(m + n)/2$ iterations.

69 Real-World Problems in the *Netlib* Suite.



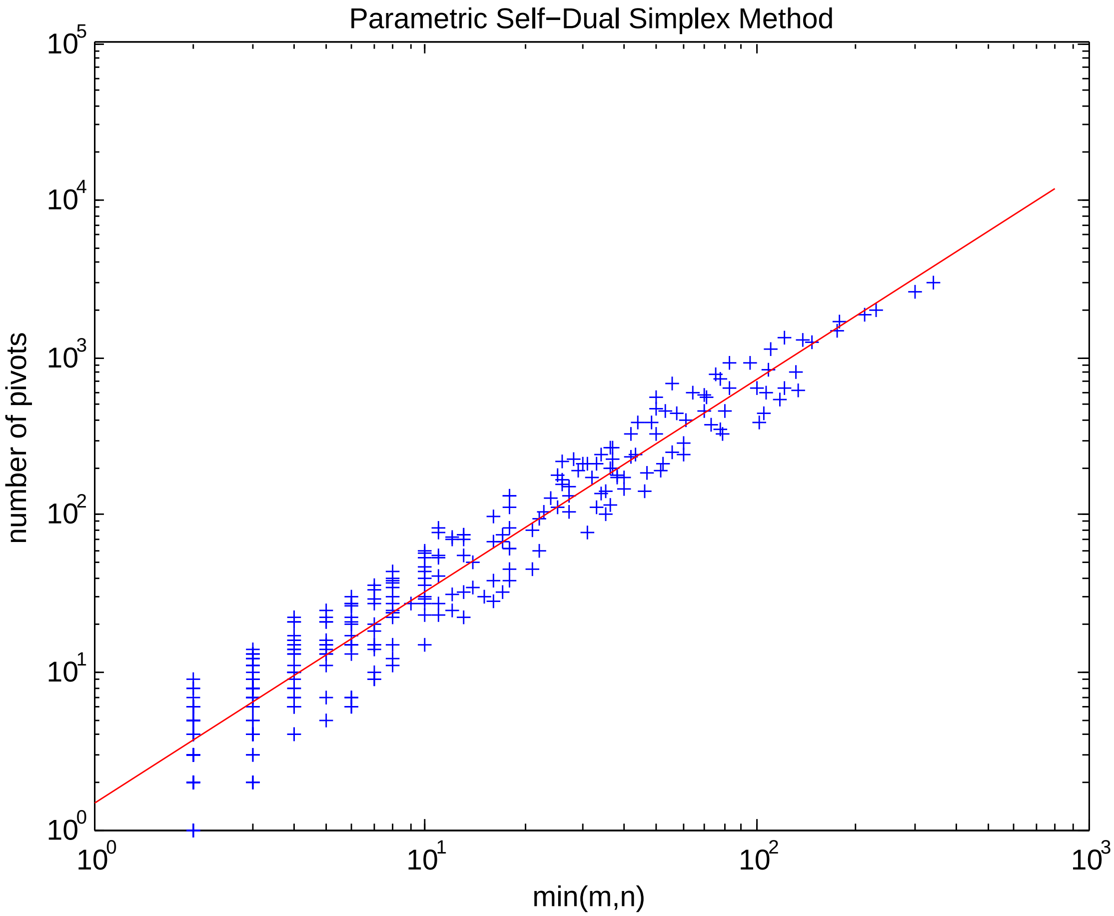
From Least Absolute Deviation: pivots $\approx 0.517(m+n)^{1.049}$

Several Randomly Generated Problems



$$\text{pivots} \;=\; 0.4165(\mathsf{m}+\mathsf{n})^{0.9759}$$

A Better Measure: $\min(m, n)$



$$\text{pivots} = 1.4880 \min(\mathsf{m}, \mathsf{n})^{1.3434}$$

Integer Programming

Gomory Cuts Method

Here's an *integer programming problem*:

Hint Undo Redo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

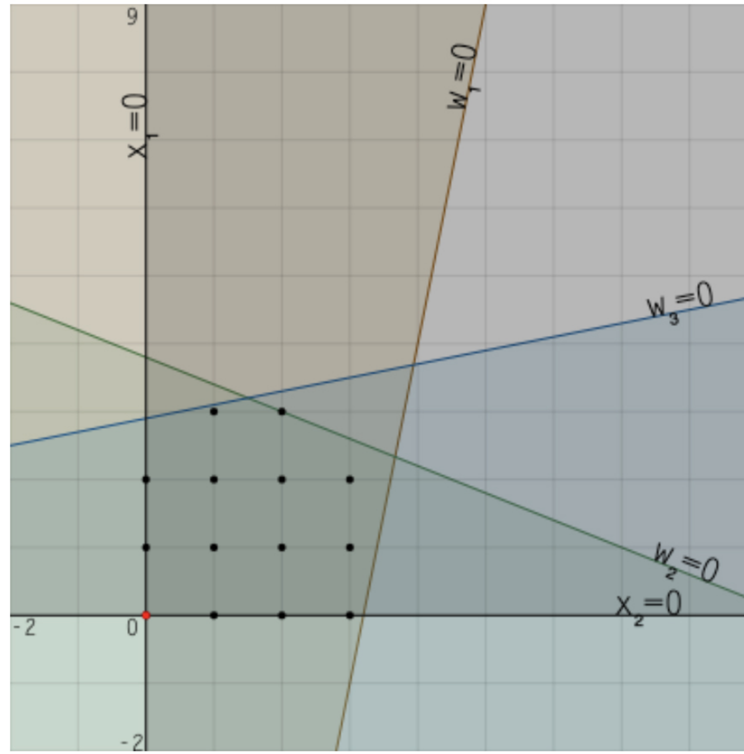
$$\text{maximize } \zeta = 8x_1 + 13x_2$$

$$w_1 = 32 - 10x_1 - (-2)x_2$$
$$w_2 = 38 - 4x_1 - 10x_2$$
$$w_3 = 29 - (-2)x_1 - 10x_2$$

$x_1, x_2 \geq 0$ integer, $w_1, w_2, w_3 \geq 0$

Add Cuts Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



LP Relaxation

Here's the solution to the *LP relaxation*:

Hint | Undo | Redo

Number format: Fraction | Zero Visibility: Visible

Current Dictionary

$$\text{maximize } \zeta = 179/3 + (-7/27)\, w_1 + (-73/54)\, w_2$$

$$x_1 = 11/3 - 5/54\, w_1 - 1/54\, w_2$$

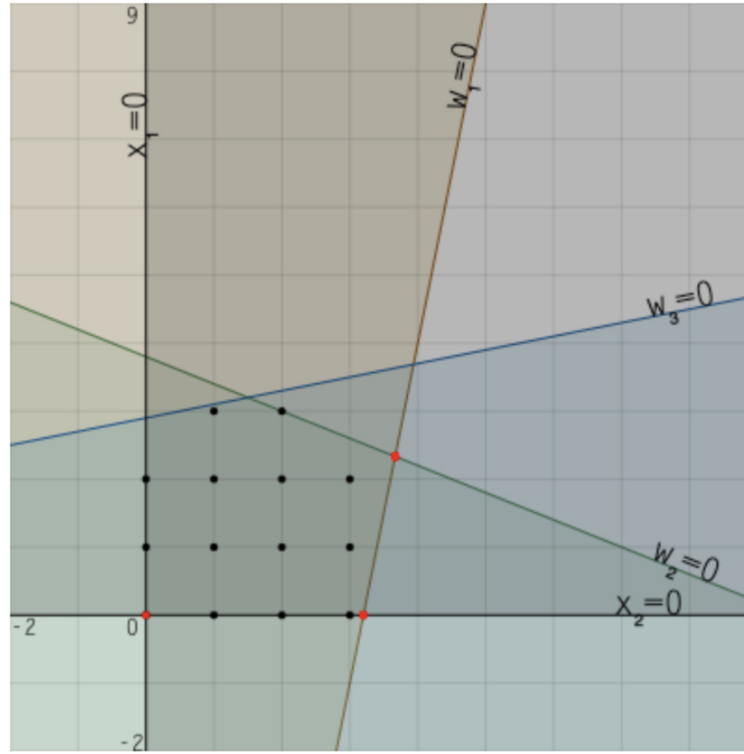
$$x_2 = 7/3 - (-1/27)\, w_1 - 5/54\, w_2$$

$$w_3 = 13 - 5/9\, w_1 - (-8/9)\, w_2$$

$x_1,\ x_2 \geq 0$ integer, $w_1,\ w_2,\ w_3 \geq 0$

Add Cuts | Optimal | Infeasible | Unbounded

Pick a judge: Bart Simpson



Neither x_1 nor x_2 are integers!

Gomory's Idea

Let's focus on x_1. It's a basic variable in the "optimal" dictionary:

$$x_1 = \frac{11}{3} - \frac{5}{54}w_1 - \frac{1}{54}w_2.$$

Let's bring the nonbasic variables over to the left-hand side:

$$x_1 + \frac{5}{54}w_1 + \frac{1}{54}w_2 = \frac{11}{3}.$$

Now, if we round down each of the coefficients on the left to the nearest smaller integer, then the left hand side will be smaller than it was. It will also be an integer whenever the variables are integer and so it will be smaller than the rounded-down value of the right-hand side:

$$x_1 + 0w_1 + 0w_2 \leq 3.$$

We want to add this as a new constraint. It will have a new slack variable, which will be a basic variable. Hence, we want to get rid of x_1 from this constraint. To do that, substitute the equation above that defines x_1 in terms of the nonbasic variables:

$$\frac{11}{3} - \frac{5}{54}w_1 - \frac{1}{54}w_2 + 0w_1 + 0w_2 \leq 3.$$

Hint Undo Redo

Number format: Fraction Zero Visibility: Visible

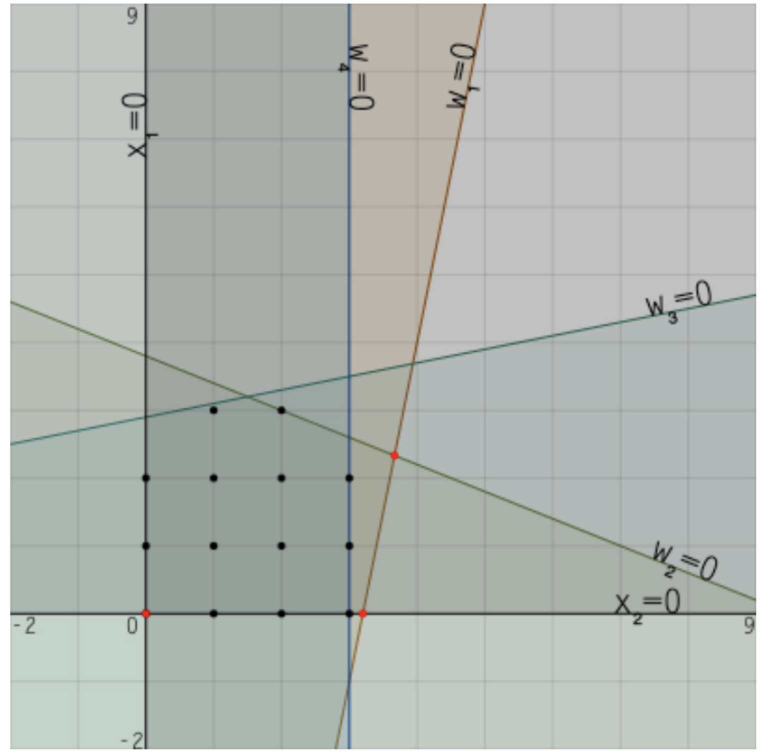
Current Dictionary

maximize ζ	=	179/3	+	-7/27	w_1	+	-73/54	w_2
x_1	=	11/3	-	5/54	w_1	-	1/54	w_2
x_2	=	7/3	-	-1/27	w_1	-	5/54	w_2
w_3	=	13	-	5/9	w_1	-	-8/9	w_2
w_4	=	-2/3	-	-5/54	w_1	-	-1/54	w_2

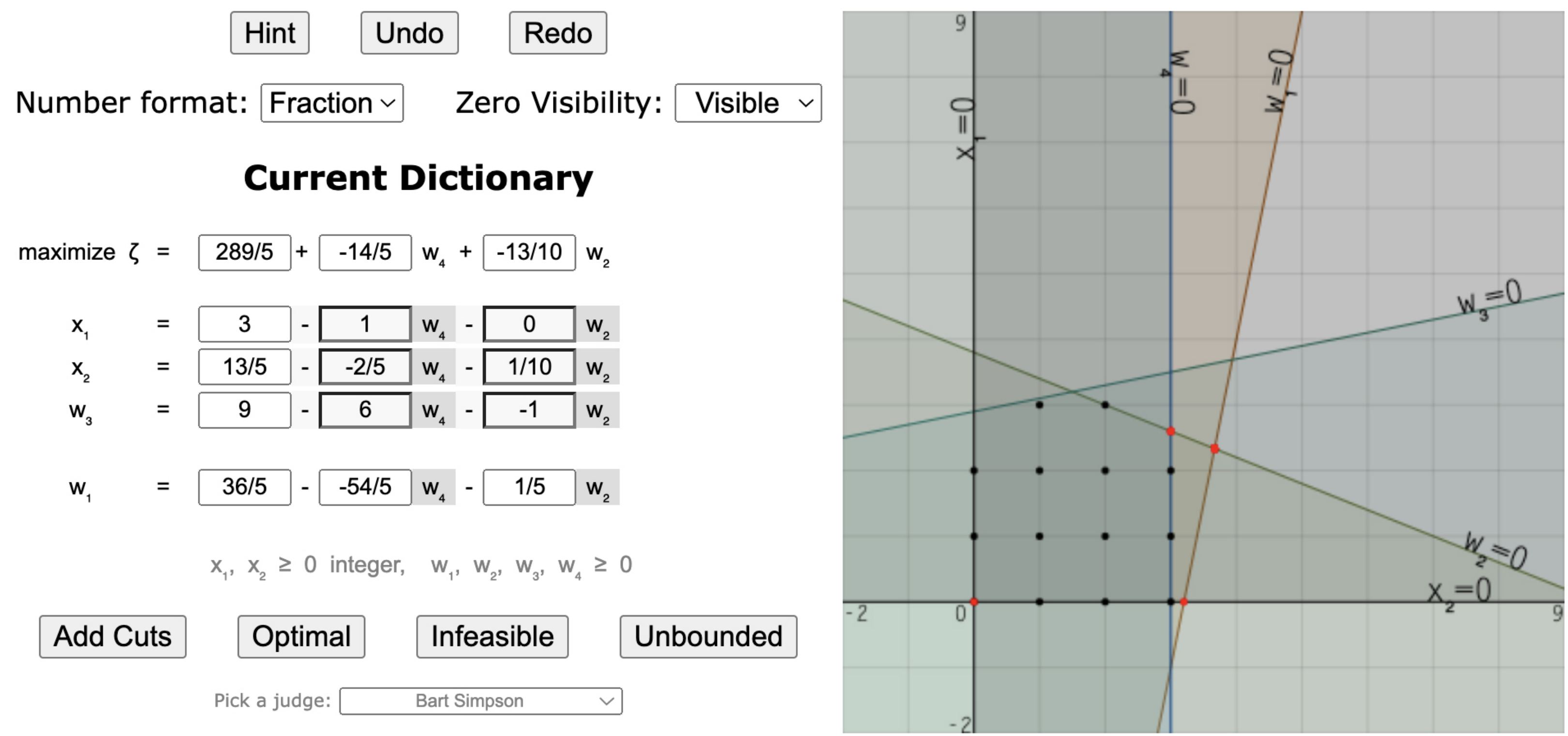
$x_1, x_2 \geq 0$ integer, $w_1, w_2, w_3, w_4 \geq 0$

Add Cuts Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



Now we just do a dual pivot to get to a new "optimal" solution:



We still have a non-integer value: $x_2 = 13/5$.

Let's make another Gomory cut:

$$x_2 - \frac{2}{5}w_4 + \frac{1}{10}w_2 = \frac{13}{5}.$$

Rounding down...

$$x_2 - w_4 + 0w_2 \leq 2$$

New constraint...

$$\frac{13}{5} + \frac{2}{5}w_4 - \frac{1}{10}w_2 - w_4 + 0w_2 \leq 2$$

Hint Undo Redo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

$$\text{maximize } \zeta = 289/5 + -14/5\, w_4 + -13/10\, w_2$$

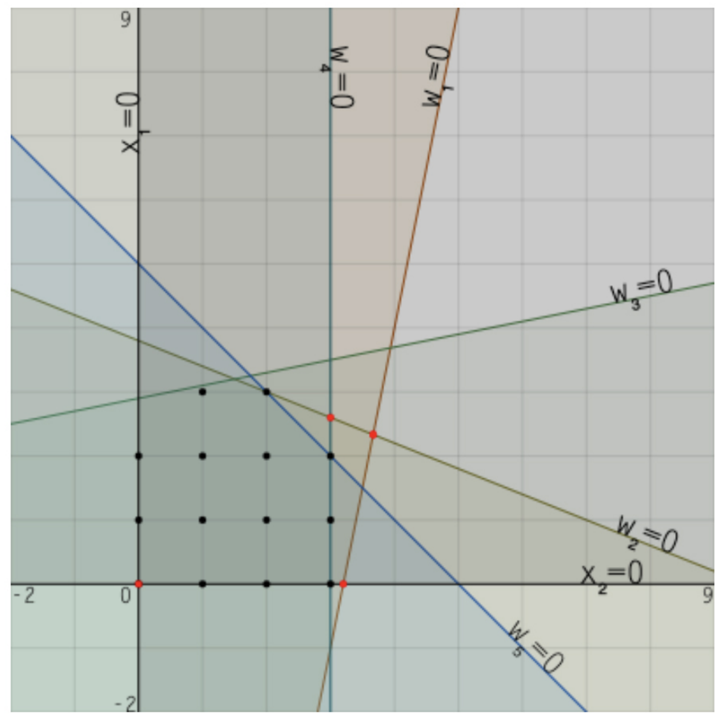
$$x_1 = 3 - 1\, w_4 - 0\, w_2$$
$$x_2 = 13/5 - -2/5\, w_4 - 1/10\, w_2$$
$$w_3 = 9 - 6\, w_4 - -1\, w_2$$

$$w_1 = 36/5 - -54/5\, w_4 - 1/5\, w_2$$

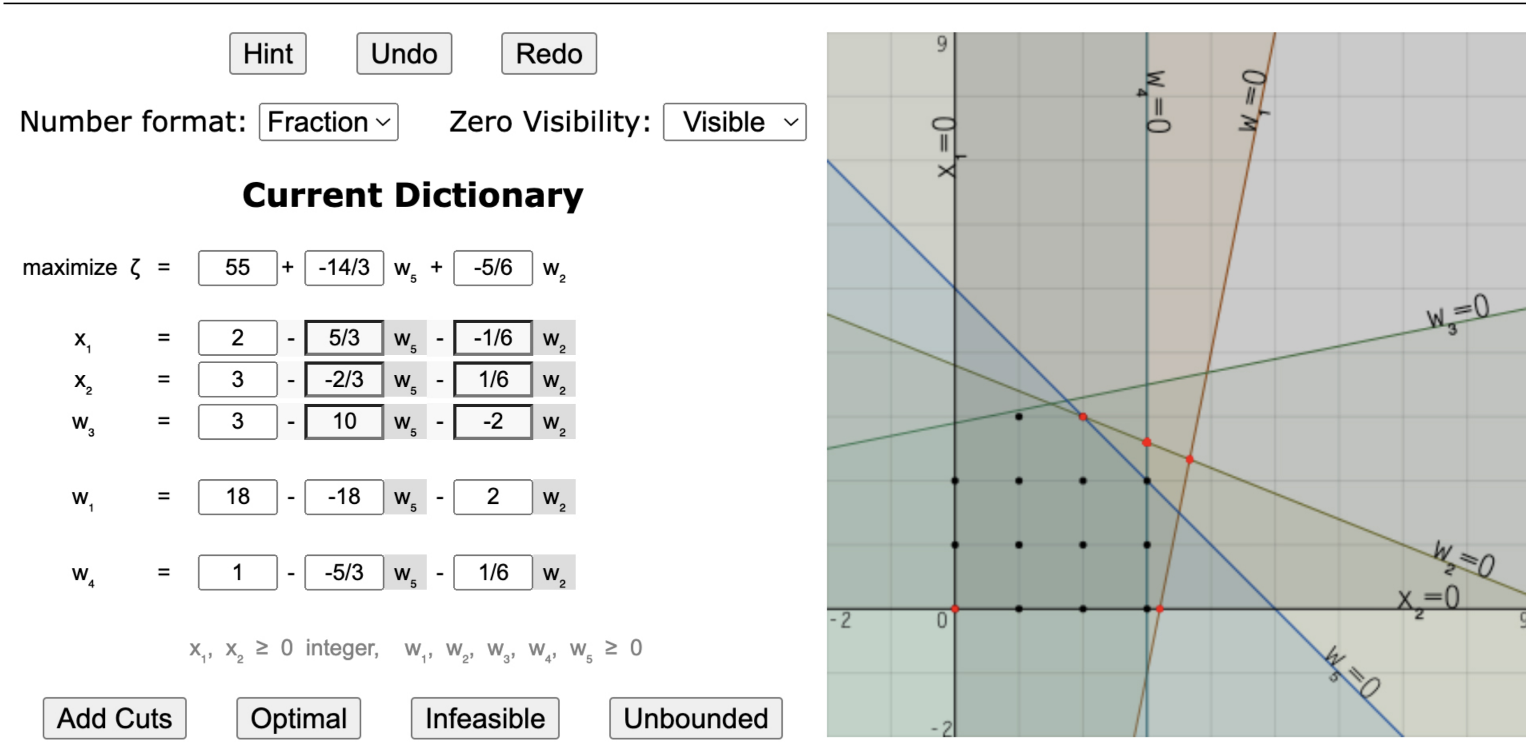
$$w_5 = -3/5 - -3/5\, w_4 - -1/10\, w_2$$

$x_1, x_2 \geq 0$ integer, $w_1, w_2, w_3, w_4, w_5 \geq 0$

Add Cuts Optimal Infeasible Unbounded



Again, we do a dual pivot:



OPTIMAL!

Real-World Applications

Real-world applications have lots of variables and lots of constraints (hundreds/thousands/more).

Hence, they can't be solved using the pivot tools.

We need software designed to solve such large problems.

Here's a website that provides online access to many options for solving such problems...

https://neos-server.org

And here's quick link to a guide explaining how to use these tools...

https://neos-guide.org

Fair Grading

SIAM Review, Vol 56, No 2, pp.337-335, 2014

Beatles University

	MAT 402	CHE 333	ANT 536	REL 101	POL 242	ECO 666
John	B−	B	B+	A−		
Paul	C+	B−		B+	A−	
George		C+	B−		B+	A−
Ringo			C+	B−	B	B+

where

MAT 402: Advanced Addition

CHE 333: Intermediate Explosives

ANT 536: The Behavior of Anthropologists

REL 101: Atheism

POL 242: Constitutional Manipulation

ECO 666: The Root Of All Evil

The Model

We assume that every grade, $g_{i,j}$ for student i in course j, can be decomposed as a sum

1. *aptitude*, a_i, of student i,
2. *easiness*, e_j, of course j,
3. plus some (presumably small) error $\epsilon_{i,j}$.

That is,

$$g_{i,j} = a_i + e_j + \epsilon_{i,j}.$$

The $g_{i,j}$'s are data. We wish to find the a_i's and the e_j's that minimize the sum of the absolute values of the $\epsilon_{i,j}$'s:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i,j} |\epsilon_{i,j}| & \\ \text{subject to} & g_{i,j} = a_i + e_j + \epsilon_{i,j} & \text{for all students/course pairs } (i,j) \\ & \displaystyle\sum_j e_j = 0. & \end{array}$$

Absolute Value Trick

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} |\epsilon_{i,j}| \\ \text{subject to} \quad & g_{i,j} = a_i + e_j + \epsilon_{i,j} \qquad \text{for all students/course pairs } (i,j) \\ & \sum_j e_j = 0. \end{aligned}$$

is equivalent to

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} t_{i,j} \\ \text{subject to} \quad & g_{i,j} - a_i - e_j \leq t_{i,j} \qquad \text{for all students/course pairs } (i,j) \\ & -t_{i,j} \leq g_{i,j} - a_i - e_j \qquad \text{for all students/course pairs } (i,j) \\ & \sum_j e_j = 0. \end{aligned}$$

The AMPL Model

```
set STUDENTS;
set COURSES;
set GRADES within {STUDENTS, COURSES};

param grade {GRADES};

var aptitude {STUDENTS};
var easiness {COURSES};
var dev {GRADES} >= 0;

minimize sum_dev: sum {(s,c) in GRADES} dev[s,c];

subject to def_pos_dev {(s,c) in GRADES:   aptitude[s] + easiness[c] - grade[s,c] <= dev[s,c];

subject to def_neg_dev {(s,c) in GRADES:   -dev[s,c] <= aptitude[s] + easiness[c] - grade[s,c];

subject to normalized_easiness:   sum {c in COURSES} easiness[c] = 0;

data;
set STUDENTS := include "names.txt" ;
set COURSES := include "courses.txt" ;
param: GRADES: grade := include "namecoursegrade.txt" ;

solve;
```

Model and three data files are here...

https://vanderbei.princeton.edu/307/models/grades/ratings.txt
https://vanderbei.princeton.edu/307/models/grades/names.txt
https://vanderbei.princeton.edu/307/models/grades/courses.txt
https://vanderbei.princeton.edu/307/models/grades/namecoursegrade.txt

Data from HARVARD's School of Science

Alex	Statistics	B-
Alex	Epistemology	B-
Alex	Phrenology	A-
Alex	Pharmacology	B
Alex	Philology	A-
Alex	Theology	A+
Alex	Geology	B+
Alex	Gynecology	A
Andy	Scientology	B-
Andy	Etymology	B-
Andy	Sociology	A-
Andy	Psychology	B
Andy	Cosmology	A-
Andy	Eulogy	A+
Andy	Immunology	B+
Andy	Methodology	A
Ariel	Epistemology	B
Ariel	Topology	B+
Ariel	Etymology	A+
Ariel	Eulogy	A+
Ariel	Genealogy	A-
Ariel	Morphology	A+
Ariel	Pathology	A-
Ariel	Technology	A+
Billy	Etymology	A+

Billy	Cosmology	A+
Billy	Philology	B
Billy	Ideology	A+
Billy	Immunology	A+
Billy	Methodology	A+
Bobby	Topology	A+
Bobby	Philology	B+
Bobby	Eulogy	A+
Bobby	Genealogy	A
Bobby	Pathology	A+
Bobby	Urology	A+
Brett	Topology	A+
Brett	Pharmacology	A-
Brett	Demonology	A+
Brett	Ecology	A+
Brett	Ideology	A+
Brett	Ontology	A
Brett	Technology	A+
Brook	Topology	A
Brook	Phrenology	B-
Brook	Cosmology	A
Brook	Philology	C+
Brook	Theology	B
Brook	Demonology	A-
Brook	Technology	A-
Brook	Terminology	A+

Cameron	Etymology	A
Cameron	Demonology	A
Cameron	Eulogy	A
Cameron	Ecology	B+
Cameron	Ontology	B
Cameron	Pathology	A
Cameron	Tautology	B+
Cameron	Technology	A-
Cary	Psychology	A+
Cary	Philology	A
Cary	Eulogy	A+
Cary	Methodology	A+
Cary	Morphology	A+
Cary	Tautology	A+
Cary	Urology	A+
Casey	Statistics	A-
Casey	Scientology	A+
Casey	Eulogy	A+
Casey	Geology	B+
Casey	Immunology	A+
Casey	Methodology	A+
Casey	Technology	A+
Casey	Terminology	A+
Chris	Topology	A+
Chris	Phrenology	A
Chris	Pharmacology	A

For full list of grades, click here.

Student-by-Student Output

```
Alex (4.18)
 2.7   3.4    Statistics (-0.8)
 2.7   3.3    Epistemology (-0.9)
 3.7   3.7    Phrenology (-0.5)
 3.0   3.7    Pharmacology (-0.5)
 3.7   3.6    Philology (-0.6)
 4.3   3.6    Theology (-0.6)
 3.3   3.3    Geology (-0.9)
 4.0   3.7    Gynecology (-0.5)
 3.43  3.54   -- avg -- (-0.64)
----------------------------------
Andy (3.02)
 2.7   3.4    Scientology (0.4)
 2.7   3.4    Etymology (0.4)
 3.7   3.4    Sociology (0.4)
 3.0   3.4    Psychology (0.4)
 3.7   3.4    Cosmology (0.4)
 4.3   3.4    Eulogy (0.4)
 3.3   3.4    Immunology (0.4)
 4.0   3.4    Methodology (0.4)
 3.43  3.44   -- avg -- ( 0.42)
----------------------------------
Ariel (3.88)
 3.0   3.0    Epistemology (-0.9)
 3.3   4.0    Topology (0.1)
 4.3   4.3    Etymology (0.4)
 4.3   4.3    Eulogy (0.4)
 3.7   4.0    Genealogy (0.1)
 4.3   4.3    Morphology (0.4)
 3.7   4.3    Pathology (0.4)
 4.3   4.0    Technology (0.1)
 3.86  4.02   -- avg -- ( 0.15)
----------------------------------

Billy (3.88)
 4.3   4.3    Etymology (0.4)
 4.3   4.3    Psychology (0.4)
 4.3   3.7    Apology (-0.2)
 4.3   4.3    Cosmology (0.4)
 3.0   3.3    Philology (-0.6)
 4.3   4.0    Ideology (0.1)
 4.3   4.3    Immunology (0.4)
 4.3   4.3    Methodology (0.4)
 4.14  4.06   -- avg -- ( 0.19)
----------------------------------
Bobby (3.88)
 4.3   4.0    Topology (0.1)
 3.3   3.3    Philology (-0.6)
 4.3   4.3    Eulogy (0.4)
 4.0   4.0    Genealogy (0.1)
 4.3   4.3    Pathology (0.4)
 4.3   4.3    Urology (0.4)
 4.08  4.03   -- avg -- ( 0.16)
----------------------------------
Brett (4.18)
 4.3   4.3    Topology (0.1)
 3.7   3.7    Pharmacology (-0.5)
 4.3   4.3    Demonology (0.1)
 4.3   4.0    Ecology (-0.2)
 4.3   4.3    Ideology (0.1)
 4.0   4.0    Ontology (-0.2)
 4.3   4.3    Technology (0.1)
 4.17  4.13   -- avg -- (-0.05)
----------------------------------

Brook (3.58)
 4.0   3.7    Topology (0.1)
 2.7   3.1    Phrenology (-0.5)
 4.0   4.0    Cosmology (0.4)
 2.3   3.0    Philology (-0.6)
 3.0   3.0    Theology (-0.6)
 3.7   3.7    Demonology (0.1)
 3.7   3.7    Technology (0.1)
 4.3   4.0    Terminology (0.4)
 3.46  3.52   -- avg -- (-0.05)
----------------------------------
Cameron (3.58)
 4.0   4.0    Etymology (0.4)
 4.0   3.7    Demonology (0.1)
 4.0   4.0    Eulogy (0.4)
 3.3   3.4    Ecology (-0.2)
 3.0   3.4    Ontology (-0.2)
 4.0   4.0    Pathology (0.4)
 3.3   3.7    Tautology (0.1)
 3.7   3.7    Technology (0.1)
 3.66  3.74   -- avg -- ( 0.16)
----------------------------------
Cary (3.88)
 4.3   4.3    Psychology (0.4)
 4.0   3.3    Philology (-0.6)
 4.3   4.3    Eulogy (0.4)
 4.3   4.3    Methodology (0.4)
 4.3   4.3    Morphology (0.4)
 4.3   4.0    Tautology (0.1)
 4.3   4.3    Urology (0.4)
 4.26  4.11   -- avg -- ( 0.24)
----------------------------------
```

Full set of output is here.

Course-by-Course Output

```
Statistics (-0.76)
 2.7   3.4    Alex (4.2)
 3.7   3.1    Casey (3.9)
 3.7   3.1    Dale (3.9)
 3.0   3.1    Darcy (3.9)
 2.7   3.1    Emerson (3.9)
 2.3   2.5    Esme (3.3)
 2.7   3.1    Harley (3.9)
 2.3   2.4    Jordan (3.2)
 4.0   3.4    Max (4.2)
 3.7   3.4    Porntip (4.2)
 2.3   2.2    Sunny (3.0)
 3.3   3.1    Sydney (3.9)
 3.03  3.01   -- avg -- ( 3.77)
-----------------------------------
Epistemology (-0.88)
 2.7   3.3    Alex (4.2)
 3.0   3.0    Ariel (3.9)
 2.7   2.7    Jade (3.6)
 2.0   2.4    Lou (3.3)
 4.0   3.3    Max (4.2)
 2.3   2.7    Robin (3.6)
 2.7   2.0    Sam (2.9)
 2.3   2.3    Tracy (3.2)
 2.71  2.71   -- avg -- ( 3.59)
-----------------------------------
Scientology (0.42)
 2.7   3.4    Andy (3.0)
 4.3   4.3    Casey (3.9)
 4.3   4.3    Dale (3.9)
 4.3   4.3    Dominique (3.9)
```

```
 3.7   3.7    Esme (3.3)
 4.3   4.5    Lindsey (4.1)
 4.3   4.3    Meryl (3.9)
 4.3   4.0    Robin (3.6)
 4.3   4.3    Sydney (3.9)
 4.06  4.13   -- avg -- ( 3.70)
-----------------------------------
Astrology (0.12)
 4.0   4.0    Darcy (3.9)
 4.3   4.3    Daryl (4.2)
 3.7   3.7    Jade (3.6)
 4.3   4.1    Morgan (4.0)
 3.0   3.1    Sunny (3.0)
 3.86  3.85   -- avg -- ( 3.73)
-----------------------------------
Topology (0.12)
 3.3   4.0    Ariel (3.9)
 4.3   4.0    Bobby (3.9)
 4.3   4.3    Brett (4.2)
 4.0   3.7    Brook (3.6)
 4.3   4.3    Chris (4.2)
 4.3   4.3    Drew (4.2)
 3.3   3.3    Jordan (3.2)
 4.0   4.0    Kelly (3.9)
 4.0   4.1    Morgan (4.0)
 4.3   4.0    Reese (3.9)
 4.3   4.3    Skye (4.2)
 4.3   4.0    Sydney (3.9)
 4.06  4.03   -- avg -- ( 3.91)
-----------------------------------
Phrenology (-0.48)
 3.7   3.7    Alex (4.2)
```

```
 2.7   3.1    Brook (3.6)
 4.0   3.7    Chris (4.2)
 4.3   3.4    Darcy (3.9)
 3.0   3.0    Devyn (3.5)
 3.3   3.1    Jade (3.6)
 4.3   3.6    Lindsey (4.1)
 3.7   3.7    Max (4.2)
 3.3   3.4    Meryl (3.9)
 3.3   3.5    Morgan (4.0)
 2.0   2.5    Sunny (3.0)
 3.7   3.4    Tanner (3.9)
 3.44  3.35   -- avg -- ( 3.82)
-----------------------------------
Pharmacology (-0.48)
 3.0   3.7    Alex (4.2)
 3.7   3.7    Brett (4.2)
 4.0   3.7    Chris (4.2)
 3.0   3.4    Kelly (3.9)
 3.7   3.7    Kim (4.2)
 4.3   3.7    Porntip (4.2)
 3.3   3.4    Sydney (3.9)
 3.57  3.61   -- avg -- ( 4.09)
-----------------------------------
Etymology (0.42)
 2.7   3.4    Andy (3.0)
 4.3   4.3    Ariel (3.9)
 4.3   4.3    Billy (3.9)
 4.0   4.0    Cameron (3.6)
 3.0   3.7    Dara (3.3)
 3.7   4.3    Harley (3.9)
 3.7   3.7    Lou (3.3)
 4.3   4.0    Peyton (3.6)
```

For full list, click here.

Portfolio Optimization

Econometrica, 71(4):1287-1297, 2003

Markowitz Shares the 1990 Nobel Prize

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

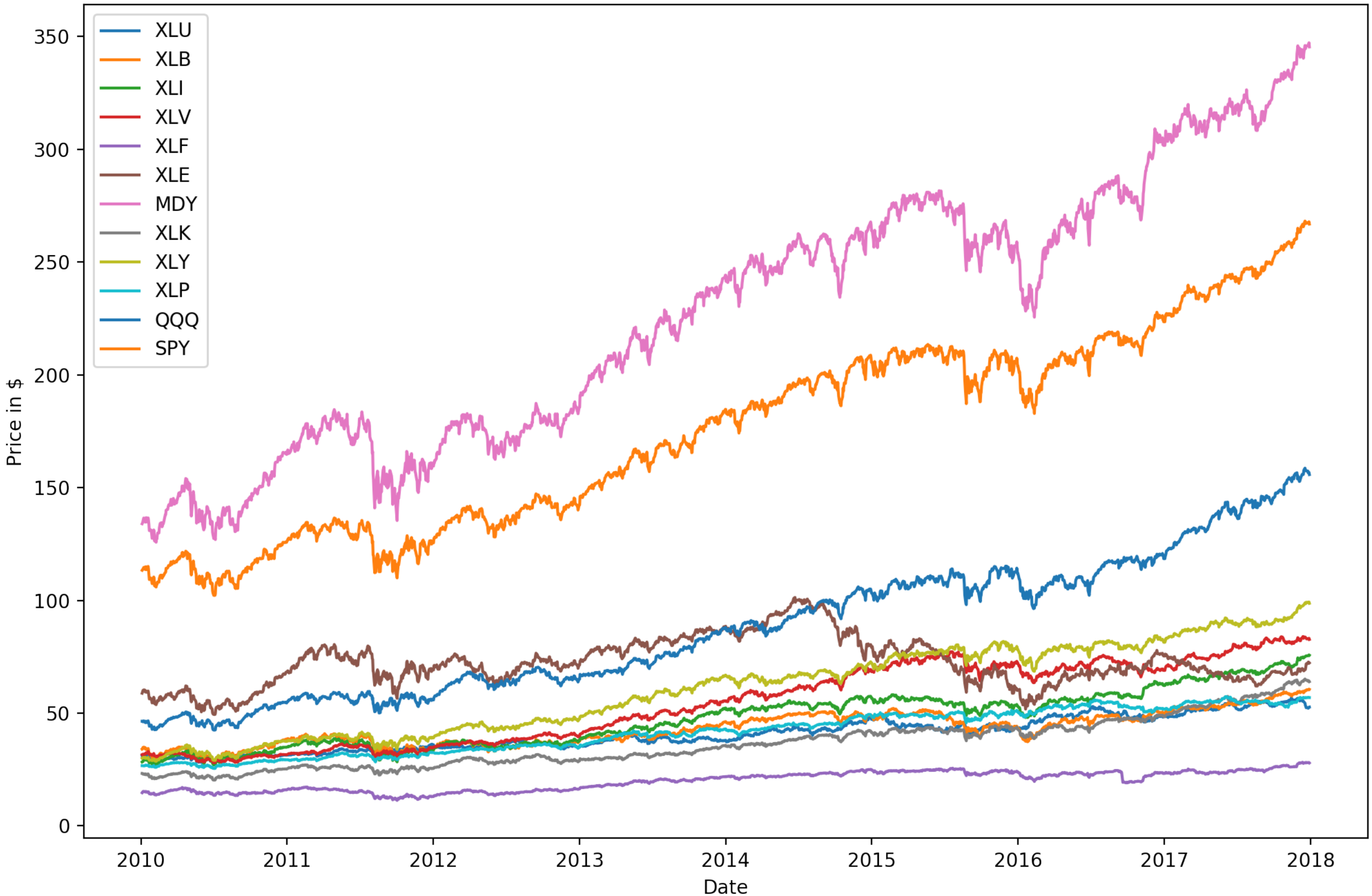
for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

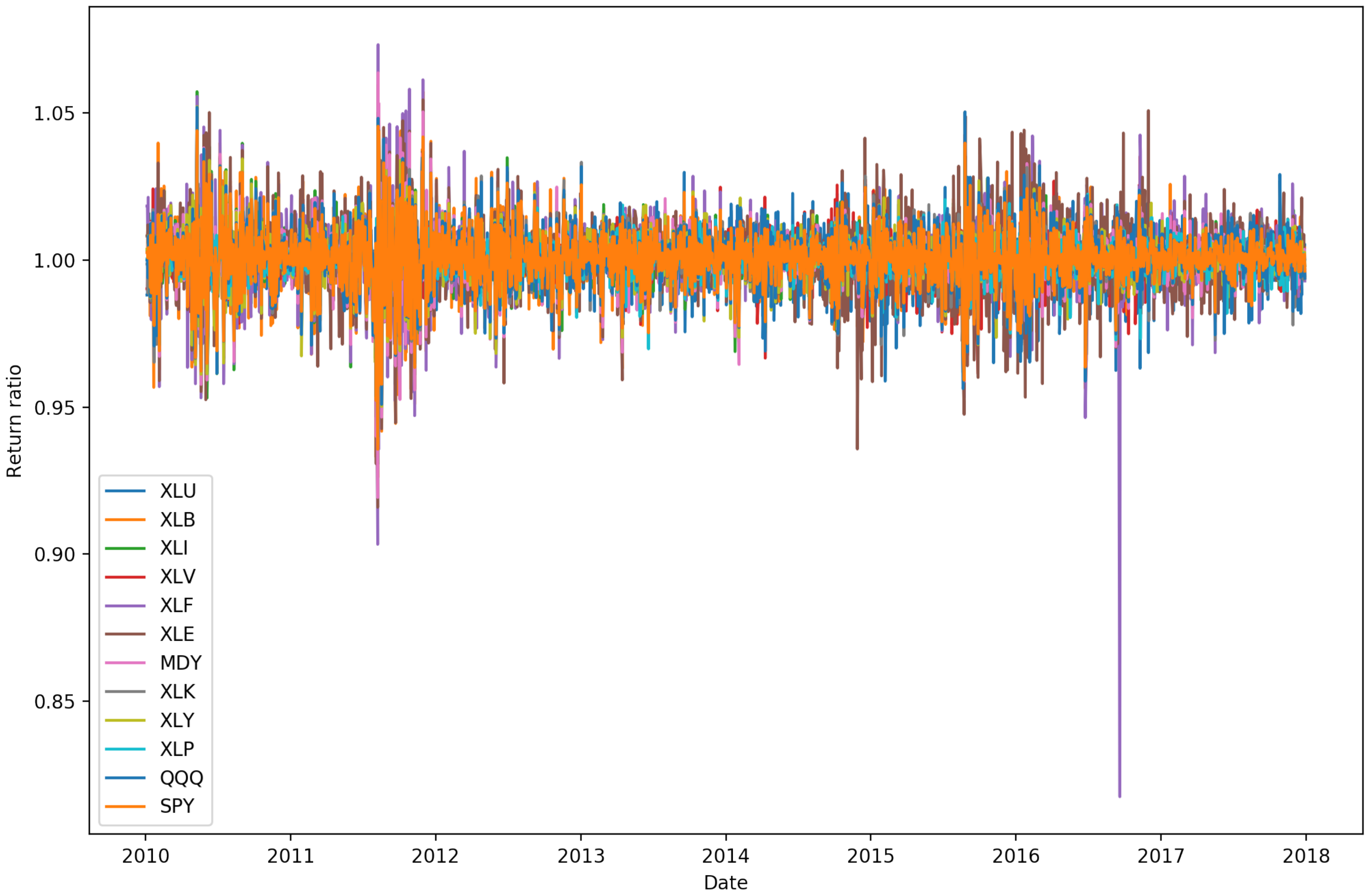
The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households' and firms' allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

Historical Data—Some ETF Prices



Notation: $S_j(t)$ = share price for investment j at time t.

Return Data: $R_j(t) = S_j(t)/S_j(t-1)$



Important observation: *volatility* is easy to see, *mean return* is lost in the noise.

Risk vs. Reward

Reward: Estimated using historical means:

$$\text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t).$$

Risk: Markowitz defined risk as the variability of the returns as measured by the historical variances:

$$\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} \left(R_j(t) - \text{reward}_j \right)^2 .$$

However, to get a linear programming problem (and for other reasons) we use the sum of the absolute values instead of the sum of the squares:

$$\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} \left| R_j(t) - \text{reward}_j \right| .$$

Why Make a Portfolio? ... Hedging

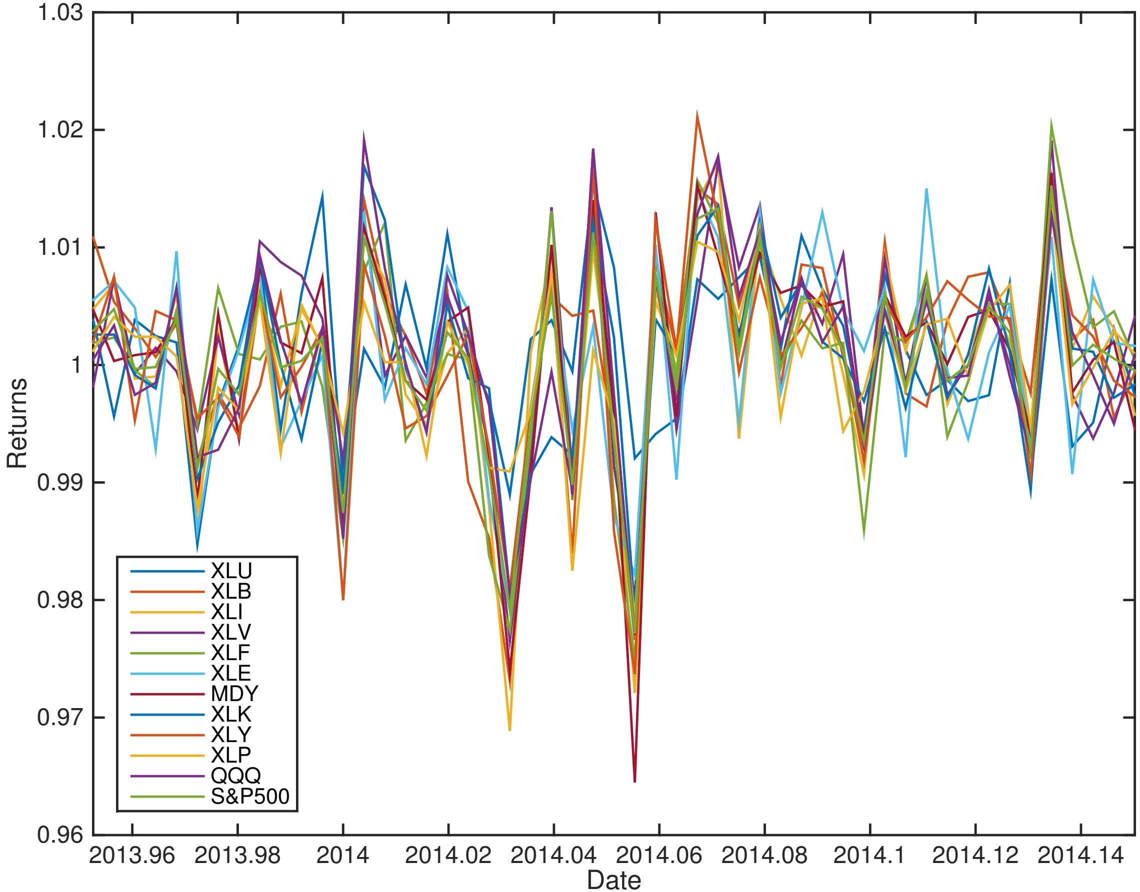
Investment A: Up 20%, down 10%, equally likely—a risky asset.

Investment B: Up 20%, down 10%, equally likely—another risky asset.

Correlation: Up-years for A are down-years for B and vice versa.

Portfolio: Half in A, half in B: up 5% every year! No risk!

Return Data: 50 days around 01/01/2014



Note: Not much *negative* correlation in price fluctuations. An up-day is an up-day and a down-day is a down-day.

Portfolios

Fractions: x_j = fraction of portfolio to invest in j

Portfolio's Historical Returns: $R_x(t) = \sum_j x_j R_j(t)$

Portfolio's Reward:

$$\begin{aligned}
\text{reward}(x) &= \frac{1}{T} \sum_{t=1}^{T} R_x(t) \\
&= \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t) \\
&= \sum_j x_j \frac{1}{T} \sum_{t=1}^{T} R_j(t) \\
&= \sum_j x_j \, \text{reward}_j
\end{aligned}$$

Portfolio's Risk:

$$
\begin{aligned}
\mathsf{risk}(x) &= \frac{1}{T}\sum_{t=1}^{T}\left(R_x(t) - \mathsf{reward}(x)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j R_j(t) - \frac{1}{T}\sum_{s=1}^{T}\sum_j x_j R_j(s)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j \left(R_j(t) - \frac{1}{T}\sum_{s=1}^{T} R_j(s)\right)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j (R_j(t) - \mathsf{reward}_j)\right)^2
\end{aligned}
$$

A Markowitz-Type Model

Decision Variables: the fractions x_j.

Objective: maximize return, minimize risk.

Fundamental Lesson: can't simultaneously optimize two objectives.

Compromise: set a lower bound μ for reward and minimize risk subject to this bound constraint:

- Parameter μ is called *reward happiness parameter*.
- Small value for μ puts emphasis on risk minimization.
- Large value for μ puts emphasis on reward maximization.

Constraints:

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) &\geq \mu \\
\sum_{j} x_j &= 1 \\
x_j &\geq 0 \qquad \text{for all } j
\end{aligned}
$$

Optimization Problem

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j} x_j(R_j(t) - \text{reward}_j)\right)^2 \\
\text{subject to} \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) \geq \mu \\
& \sum_{j} x_j = 1 \\
& x_j \geq 0 \qquad \text{for all } j
\end{aligned}
$$

AMPL: Model

```
set Assets;  # asset categories
set Dates;   # dates

param T := card(Dates);
param mu;
param R {Dates,Assets};
param mean {j in Assets} := ( sum{t in Dates} R[t,j] )/T;
param Rdev {t in Dates, j in Assets} := R[t,j] - mean[j];
param variance {j in Assets} := ( sum{t in Dates} Rdev[t,j]^2 )/T;

var x{Assets} >= 0;

minimize risk: sum{t in Dates} (sum{j in Assets} Rdev[t,j]*x[j])^2 / T;

s.t. reward_bound: sum{j in Assets} mean[j]*x[j] >= mu;
s.t. tot_mass: sum{j in Assets} x[j] = 1;
```

AMPL: Data

```
data;

set Assets := mdy xlb xli xlu spy qqq xle xlk xlv xlf xlp xly;
set Dates := include 'dates';

param R: mdy xlb xli xlu spy qqq xle xlk xlv xlf xlp xly :=
  include 'returns.data' ;

printf {j in Assets}: "%10.7f %10.5f \n",
       mean[j]^(12), sum{t in Dates} (Rdev[t,j])^2/T > "assets";
```

AMPL: Solve, and Print

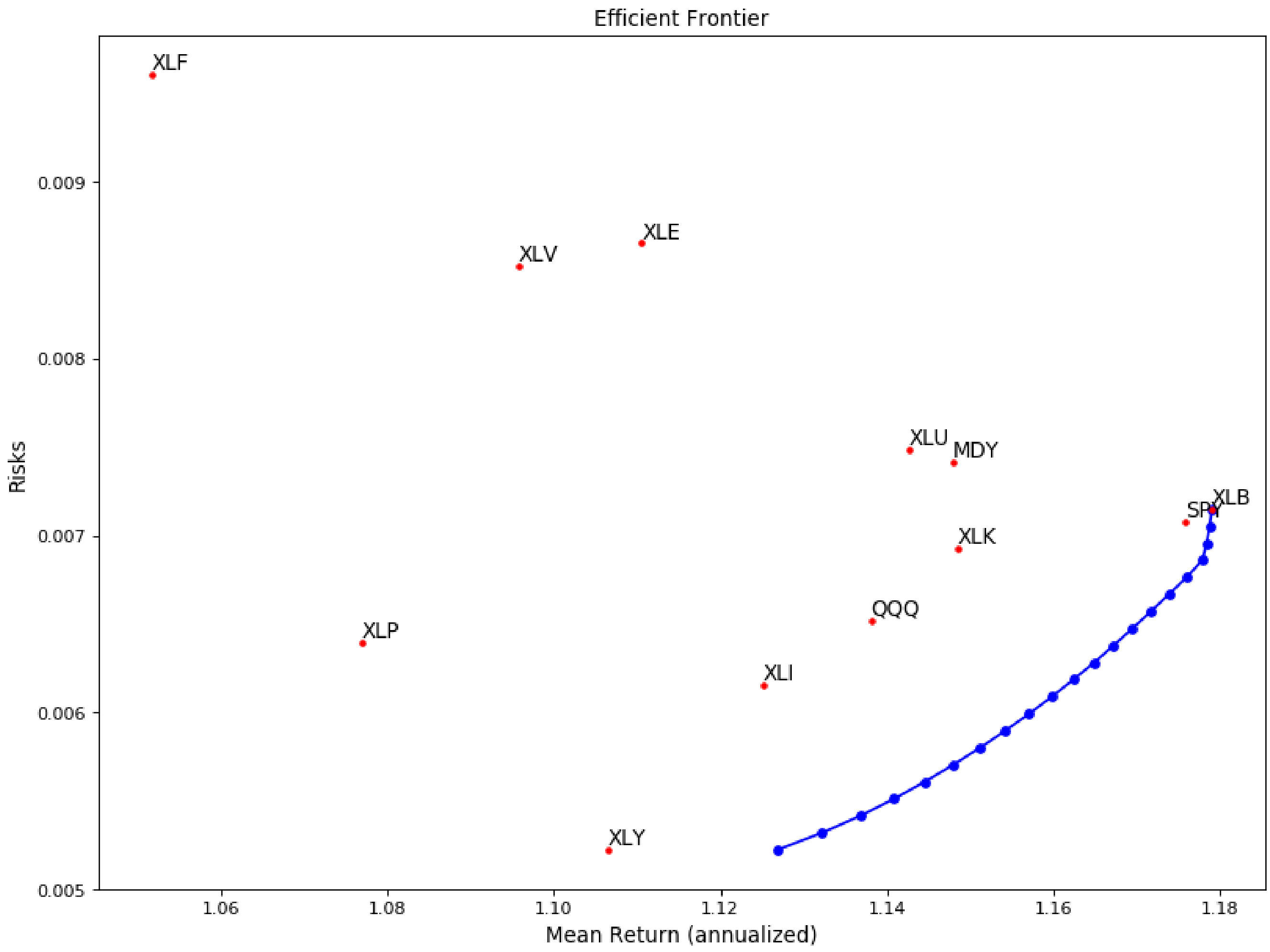
```
set assets_min_var ordered := {j in Assets: variance[j] == min {jj in Assets} variance[jj]};
param maxmean := max {j in Assets} mean[j];
param minmean := mean[first(assets_min_var)];

display mean, variance;
display minmean, maxmean;

printf {j in Assets}: " %5s  ", j > "portfolio_minrisk";
printf " |  reward   risk \n" > "portfolio_minrisk";
for {k in 0..20} {
  display k;
  let mu := (k/20)*minmean + (1-k/20)*maxmean;

  solve;

  printf {j in Assets}: "%7.4f ", x[j] > "portfolio_minrisk";
  printf " | %7.4f %7.4f \n",
      (sum{j in Assets} mean[j]*x[j])^(12),
      sum{t in Dates} (sum{j in Assets} Rdev[t,j]*x[j])^2 / T
              > "portfolio_minrisk";
}
```

Efficient Frontier

Varying risk bound μ produces the so-called *efficient frontier*.
Portfolios on the efficient frontier are reasonable.
Portfolios not on the efficient frontier can be strictly improved.

XLU	XLB	XLI	XLV	XLF	XLE	MDY	XLK	XLY	XLP	QQQ	SPY	Risk	Reward
	1.00000											0.00715	1.00063
	0.91073										0.08927	0.00705	1.00063
	0.80327										0.19673	0.00696	1.00063
	0.64003										0.35997	0.00686	1.00063
	0.52089									0.03862	0.44049	0.00676	1.00062
	0.50041							0.01272		0.06919	0.41768	0.00667	1.00062
	0.48484							0.04132		0.07129	0.40254	0.00657	1.00061
	0.46483							0.06857		0.07658	0.39002	0.00647	1.00060
	0.44030							0.09633		0.08232	0.38105	0.00638	1.00059
	0.42825							0.12917		0.08171	0.36086	0.00628	1.00059
	0.39737							0.16114		0.08506	0.35643	0.00619	1.00058
	0.36890							0.19318		0.09133	0.34659	0.00609	1.00057
	0.33802							0.22223	0.00451	0.09494	0.34030	0.00599	1.00056
	0.29959							0.23687	0.01707	0.10664	0.33984	0.00590	1.00055
	0.27975							0.26587	0.02543	0.10951	0.31943	0.00580	1.00054
	0.25688							0.28212	0.03974	0.12461	0.29666	0.00570	1.00053
	0.24677							0.30348	0.05438	0.13634	0.25903	0.00561	1.00052
	0.23570							0.32960	0.07273	0.13670	0.22527	0.00551	1.00051
	0.21978							0.36630	0.09093	0.12719	0.19580	0.00541	1.00049
	0.21069							0.40713	0.10881	0.12695	0.14641	0.00532	1.00048
	0.18010							0.46128	0.12077	0.13760	0.10025	0.00522	1.00046

Efficient Frontier



Downloading the AMPL model and data

AMPL Model:

https://vanderbei.princeton.edu/307/ampl/markL2_minrisk.txt

List of dates:

https://vanderbei.princeton.edu/307/ampl/dates.txt

Monthly return data:

https://vanderbei.princeton.edu/307/ampl/returns.txt

Data from

Yahoo Groups Finance

Alternative Formulation

Minimize risk using *least absolute deviations* as the risk measure:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j x_j(R_j(t) - \text{reward}_j)\right| \\
\text{subject to} & \displaystyle \frac{1}{T}\sum_{t=1}^{T}\sum_j x_j R_j(t) \geq \mu \\
& \displaystyle \sum_j x_j = 1 \\
& x_j \geq 0 \qquad \text{for all } j
\end{array}
$$

Because of absolute values not a linear programming problem.

Easy to convert...

Main Idea For The Conversion

Using the “greedy substitution”, we introduce new variables to represent the troublesome part of the problem

$$y_t = \left| \sum_j x_j(R_j(t) - \text{reward}_j) \right|$$

to get

$$\begin{aligned} \text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} y_t \\ \text{subject to} \quad & \left| \sum_j x_j(R_j(t) - \text{reward}_j) \right| = y_t \qquad \text{for all } t \\ & \frac{1}{T}\sum_{t=1}^{T}\sum_j x_j R_j(t) \geq \mu \\ & \sum_j x_j = 1 \\ & x_j \geq 0 \qquad \text{for all } j \end{aligned}$$

We then note that the constraint defining y_t can be relaxed to a pair of inequalities:

$$-y_t \leq \sum_j x_j(R_j(t) - \text{reward}_j) \leq y_t.$$

A Linear Programming Formulation

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\frac{1}{T}\sum_{t=1}^{T} y_t & \\
\text{subject to} & \displaystyle -y_t \le \sum_j x_j (R_j(t) - \text{reward}_j) \le y_t & \text{for all } t \\
& \displaystyle \frac{1}{T}\sum_{t=1}^{T}\sum_j x_j R_j(t) \ge \mu & \\
& \displaystyle \sum_j x_j = 1 & \\
& x_j \ge 0 & \text{for all } j
\end{array}
$$