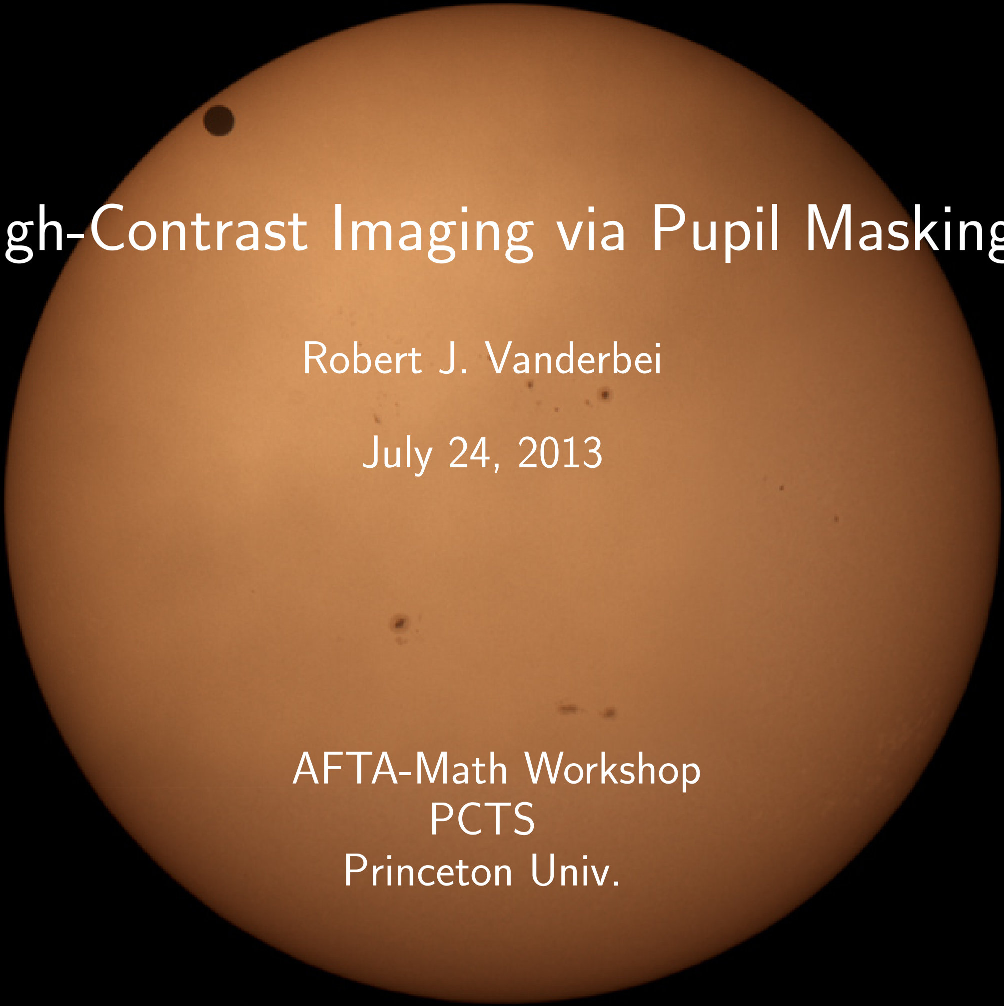

High-Contrast Imaging via Pupil Masking

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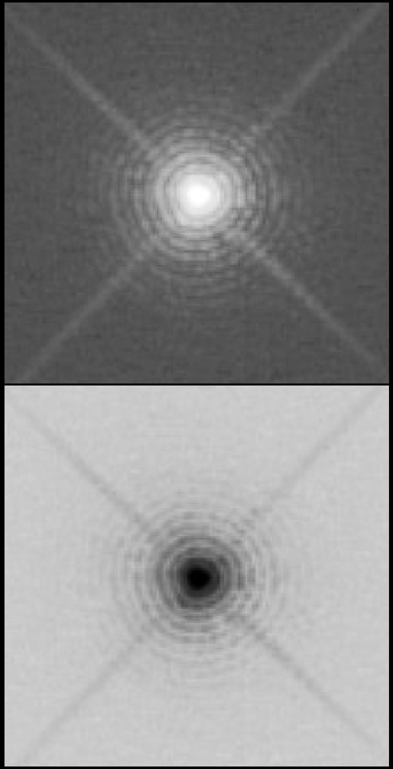
AFTA-Math Workshop
PCTS
Princeton Univ.

http://www.princeton.edu/∼rvdb

High-Contrast Imaging for Planet-Finding

Build a telescope capable of finding Earth-like planets around nearby Sun-like stars.

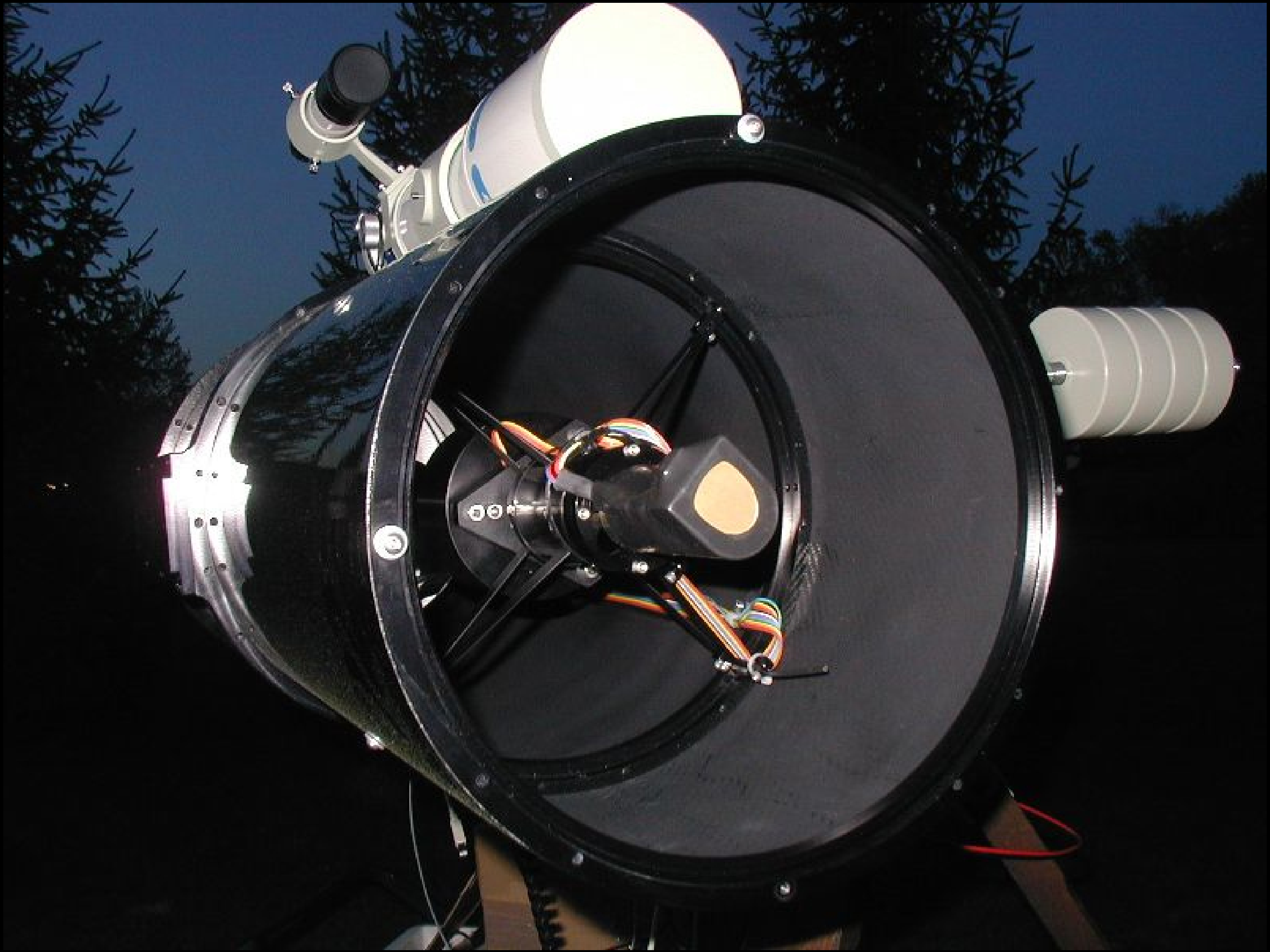


Problem is hard:

- Star is 10^{10} times brighter than the planet.
- Angular separation is small ≈ 0.05 arcseconds.
- Light is a wave: the star is not a pinpoint of light—it has a *diffraction pattern*.
- Light is particles (photons) $\Rightarrow$ Poisson statistics.

The diffraction pattern is the magnitude-squared of the *Fourier transform* of the telescope's *pupil*.

Pupil Masking/Apodization



Pupil Masking/Apodization

Fourier/Fresnel Propagation

We start with the Fresnel propogation formula for propogating an electric field a distance f after a lens/mirror of focal length f:

$$E_{\text{out}}(\xi,\eta) = \frac{i}{\lambda f} e^{\frac{\pi i}{\lambda f}(\xi^2+\eta^2)} \iint e^{-\frac{2\pi i}{\lambda f}(x\xi+y\eta)} A(x,y) E_{\text{in}}(x,y)\,dy\,dx.$$

Here,

- E_{in} denotes the *electromagnetic field* at the *pupil*;
- E_{out} denotes the *electromagnetic field* at the *image*;
- A denotes the *apodization*;
- Temporal variations are suppressed.

At the design stage, we can

- Assume that the input field E_{in} is an on-axis plane wave;
- Normalize so that $E_{\text{in}} \equiv 1$;
- Choose units to be *aperture*'s in the pupil plane and λ/D in the image plane;
- Drop leading unit magnitude factors.

Result: a simple *2D Fourier transform*.

Fourier Propagation in Simple Units

Let $f(x, y)$ denote the transmissivity (i.e., *apodization*) at location (x, y) on the surface of a filter placed over the pupil of a telescope.

The *electromagnetic field* in the image plane of such a telescope associated with an on-axis point source (i.e., a star) is proportional to the 2D Fourier transform of the apodization f:

$$\widehat{f}(\xi, \eta) = \iint e^{2\pi i (x\xi + y\eta)} f(x, y) dx dy.$$

The *intensity* of the light in the image is proportional to the *magnitude squared* of $\widehat{f}$.

Assuming that the underlying telescope has a circular opening of radius one half, we impose the following constraint on f:

$$f(x, y) = 0 \quad \text{for} \quad x^2 + y^2 > 1/2^2.$$

Optimized Apodizations

Maximize *light throughput* subject to constraint that almost no light reaches a given *dark zone* $\mathcal{D}$ and other *structural* constraints:

$$
\begin{array}{lrcll}
\text{maximize} & \multicolumn{3}{l}{\displaystyle\iint_{\square} f(x,y)\,dx\,dy \qquad \left(\; = \; \widehat{f}(0,0) \; \right)} & \\
\text{subject to} & & \left|\widehat{f}(\xi,\eta)\right| \leq \varepsilon\, \widehat{f}(0,0), & & (\xi,\eta) \in \mathcal{D}, \\
 & & f(x,y) = 0, & & x^2 + y^2 > 1/2^2, \\
 & 0 \leq & f(x,y) \leq 1, & & \text{for all } x, y.
\end{array}
$$

Here, ε is a small positive constant (on the order of 10^{-5}).

In general, the Fourier transform $\widehat{f}$ is complex valued.

This optimization problem has a *linear objective* function and both *linear* and *second-order cone* constraints.

Hence, a discretized version can be solved (to a *global optimum*).

Exploiting Symmetry

Assuming that the filter can be symmetric with respect to reflection about both axes (note: sometimes not possible), the Fourier transform can be written as

$$\widehat{f}(\xi, \eta) = 4 \int_0^{1/2} \int_0^{1/2} \cos(2\pi x \xi) \cos(2\pi y \eta) f(x, y) dx dy.$$

In this case, the Fourier transform is real and so the second-order cone constraints can be replaced with a pair of inequalities,

$$-\varepsilon\ \widehat{f}(0,0) \leq \widehat{f}(\xi, \eta) \leq \varepsilon\ \widehat{f}(0,0),$$

making the problem an *infinite dimensional linear programming problem*.

Note: *Curse of Dimensionality*

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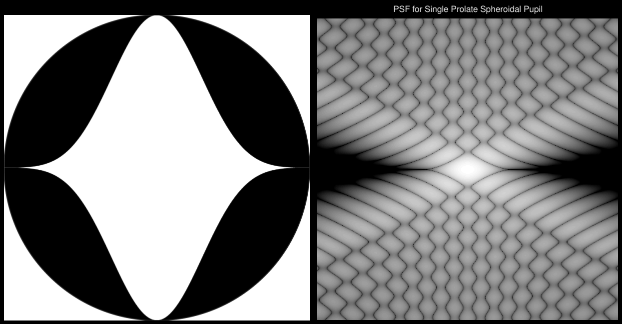
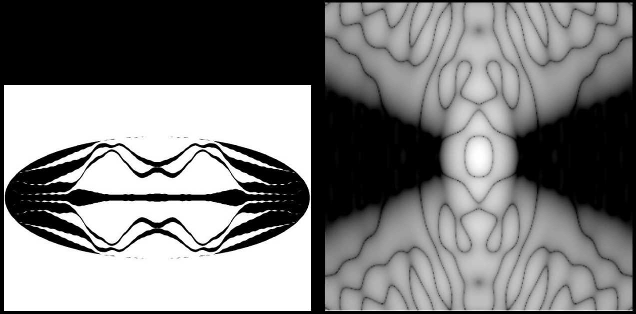
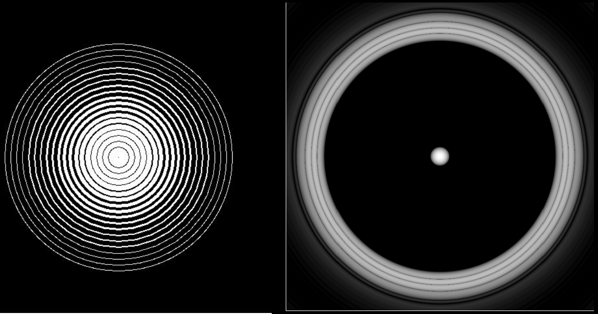
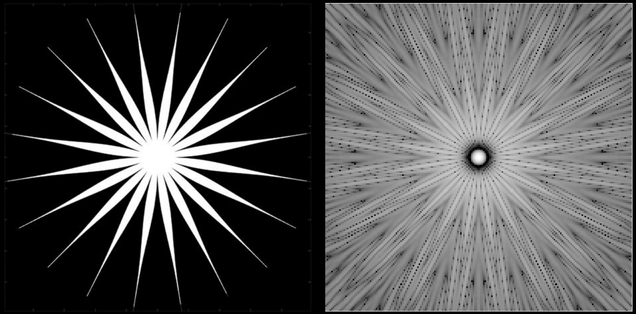
Note: *Curse of Dimensionality:* $2 > 1$.

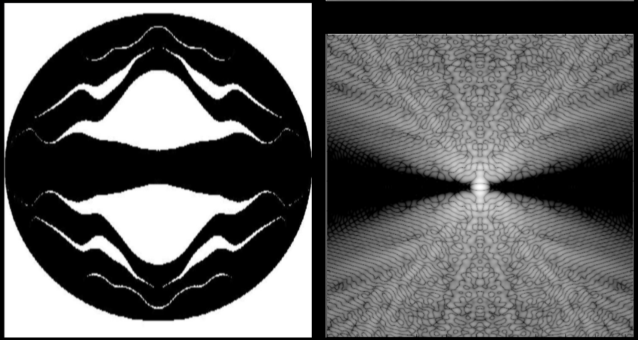
Potpourri of Pupil Masks











Discretization

Consider a two-dimensional Fourier transform

$$\widehat{f}(\xi,\eta) = 4\int_0^{1/2}\int_0^{1/2} \cos(2\pi x\xi)\cos(2\pi y\eta) f(x,y)dxdy.$$

Its discrete approximation can be computed as

$$\widehat{f}_{j_1,j_2} = 4\sum_{k_2=1}^{n}\sum_{k_1=1}^{n} \cos(2\pi x_{k_1}\xi_{j_1})\cos(2\pi y_{k_2}\eta_{j_2}) f_{k_1,k_2}\Delta x\Delta y, \quad 1 \le j_1, j_2 \le m,$$

where

$$\begin{aligned}
x_k &= (k-1/2)\Delta x, & 1 \le k \le n,\\
y_k &= (k-1/2)\Delta y, & 1 \le k \le n,\\
\xi_j &= (j-1/2)\Delta\xi, & 1 \le j \le m,\\
\eta_j &= (j-1/2)\Delta\eta, & 1 \le j \le m,\\
f_{k_1,k_2} &= f(x_{k_1}, y_{k_2}), & 1 \le k_1, k_2 \le n,\\
\widehat{f}_{j_1,j_2} &\approx \widehat{f}(\xi_{j_1}, \eta_{j_2}), & 1 \le j_1, j_2 \le m.
\end{aligned}$$

Complexity: m^2n^2.

A Clever (and Trivial!) Idea

The obvious brute force calculation requires m^2n^2 operations.

However, we can “factor” the double sum into a nested pair of sums.

Introducing new variables that represent the inner sum, we get:

$$\begin{aligned} g_{j_1,k_2} &= 2\sum_{k_1=1}^{n} \cos(2\pi x_{k_1}\xi_{j_1}) f_{k_1,k_2}\Delta x, && 1 \le j_1 \le m,\ \ 1 \le k_2 \le n, \\ \widehat{f}_{j_1,j_2} &= 2\sum_{k_2=1}^{n} \cos(2\pi y_{k_2}\eta_{j_2}) g_{j_1,k_2}\Delta y, && 1 \le j_1, j_2 \le m, \end{aligned}$$

Formulated this way, the calculation requires only $mn^2 + m^2n$ operations.

Brute Force vs Clever Approach

On the following page we show two AMPL model formulations of this problem.

On the left is the version expressed in the straightforward one-step manner.

On the right is the AMPL model for the same problem but with the Fourier transform expressed as a pair of transforms—the so-called *two-step process*.

The dark zone $\mathcal{D}$ is a pair of sectors of an annulus with inner radius 4 and outer radius 20.

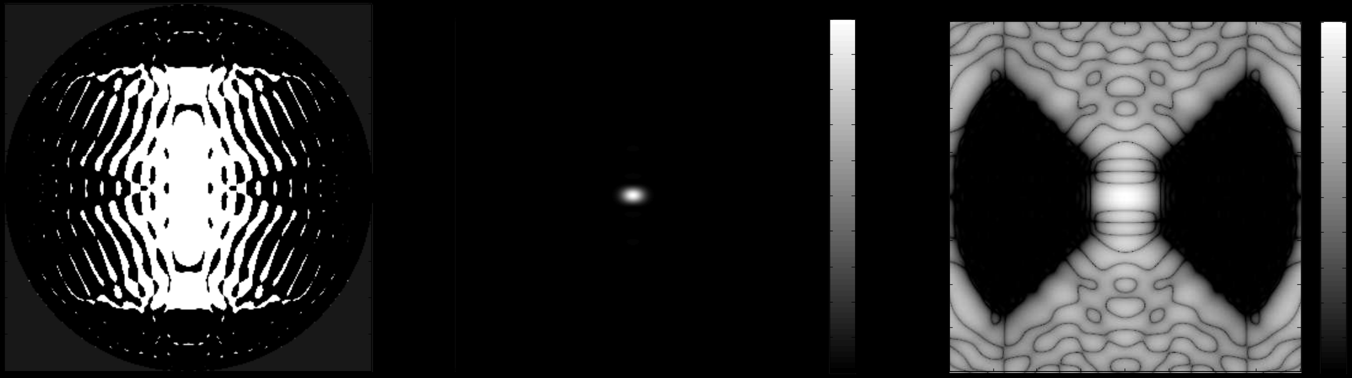
Except for the resolution, the two models produce the same result.

Two AMPL Models

```
param rho0 := 4;       param rho1 := 20;
param m := 35;     # discretization parameter
param n := 150;    # discretization parameter
param dx := 1/(2*n);   param dy := dx;

set Xs := setof {j in 0.5..n-0.5 by 1} j/(2*n);
set Ys := Xs;
set Pupil :=
    setof {x in Xs, y in Ys: x^2+y^2<0.25} (x,y);
set Xis  := setof {j in 0..m} j*rho1/m;
set Etas := Xis;
set DarkHole := setof {xi in Xis, eta in Etas:
                        xi^2+eta^2>=rho0^2 &&
                        xi^2+eta^2<=rho1^2 &&
                        eta <= xi } (xi,eta);

var f {(x,y) in Pupil} >= 0, <= 1;

var fhat {xi in Xis, eta in Etas};

maximize area: sum {(x,y) in Pupil} f[x,y]*dx*dy;




subject to fhat_def {xi in Xis, eta in Etas}:
    fhat[xi,eta] = 4*sum {(x,y) in Pupil}
        f[x,y]*cos(2*pi*x*xi)
              *cos(2*pi*y*eta)*dx*dy;
subject to sidelobe_pos {(xi,eta) in DarkHole}:
    fhat[xi,eta] <= 10^(-5)*fhat[0,0];
subject to sidelobe_neg {(xi,eta) in DarkHole}:
    -10^(-5)*fhat[0,0] <= fhat[xi,eta];

solve;
```

```
param rho0 := 4;       param rho1 := 20;
param m := 35;     # discretization parameter
param n := 1000;   # discretization parameter
param dx := 1/(2*n);   param dy := dx;

set Xs := setof {j in 0.5..n-0.5 by 1} j/(2*n);
set Ys := Xs;
set Pupil :=
    setof {x in Xs, y in Ys: x^2+y^2 < 0.25} (x,y);
set Xis  := setof {j in 0..m} j*rho1/m;
set Etas := Xis;
set DarkHole := setof {xi in Xis, eta in Etas:
                        xi^2+eta^2>=rho0^2 &&
                        xi^2+eta^2<=rho1^2 &&
                        eta <= xi } (xi,eta);

var f {(x,y) in Pupil} >= 0, <= 1;
var g {xi in Xis, y in Ys};
var fhat {xi in Xis, eta in Etas};

maximize area: sum {(x,y) in Pupil} f[x,y]*dx*dy;

subject to g_def {xi in Xis, y in Ys}:
    g[xi,y] = 2*sum {x in Xs: (x,y) in Pupil}
        f[x,y]*cos(2*pi*x*xi)*dx;
subject to fhat_def {xi in Xis, eta in Etas}:
    fhat[xi,eta] = 2*sum {y in Ys}
        g[xi,y]*cos(2*pi*y*eta)*dy;

subject to sidelobe_pos {(xi,eta) in DarkHole}:
    fhat[xi,eta] <= 10^(-5)*fhat[0,0];
subject to sidelobe_neg {(xi,eta) in DarkHole}:
    -10^(-5)*fhat[0,0] <= fhat[xi,eta];

solve;
```

Optimal Solution



Left. The optimal apodization found by either of the models shown on previous slide.

Center. Plot of the star's image (using a linear stretch).

Right. Logarithmic plot of the star's image (black $= 10^{-10}$).

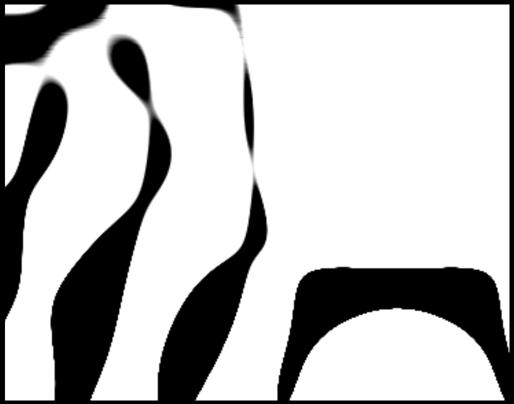
Notes:

- The "apodization" turns out to be purely opaque and transparent (i.e., a mask).
- The mask has "islands" and therefore must be laid on glass.

Close Up

Brute force with $n = 150$

Two-step with $n = 1000$



Summary Problem Stats

Comparison between a few sizes of the one-step and two-step models.

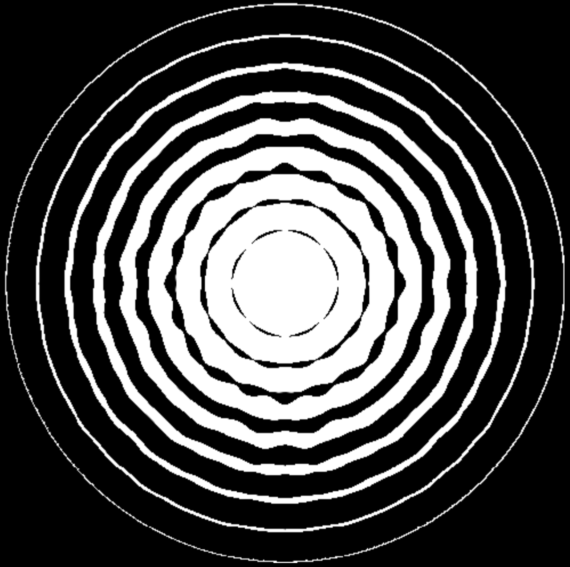
Problem-specific stats.

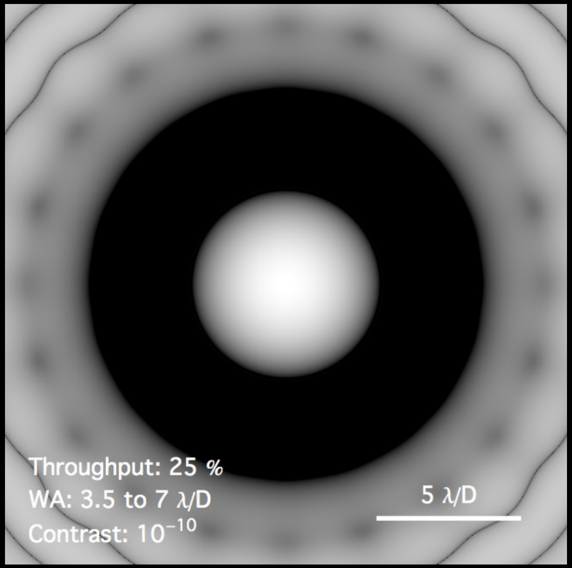
Model	n	m	constraints	variables	nonzeros	arith. ops.
One step	150	35	976	17,672	17,247,872	17,196,541,336
One step	250	35	*	*	*	*
Two step	150	35	7,672	24,368	839,240	3,972,909,664
Two step	500	35	20,272	215,660	7,738,352	11,854,305,444
Two step	1000	35	38,272	822,715	29,610,332	23,532,807,719

Hardware/Solution-specific performance comparison data.

Model	n	m	iterations	primal objective	dual objective	cpu time (sec)
One step	150	35	54	0.05374227247	0.05374228041	1380
One step	250	35	*	*	*	*
Two step	150	35	185	0.05374233071	0.05374236091	1064
Two step	500	35	187	0.05395622255	0.05395623990	4922
Two step	1000	35	444	0.05394366337	0.05394369256	26060

Concentric Ring Masks

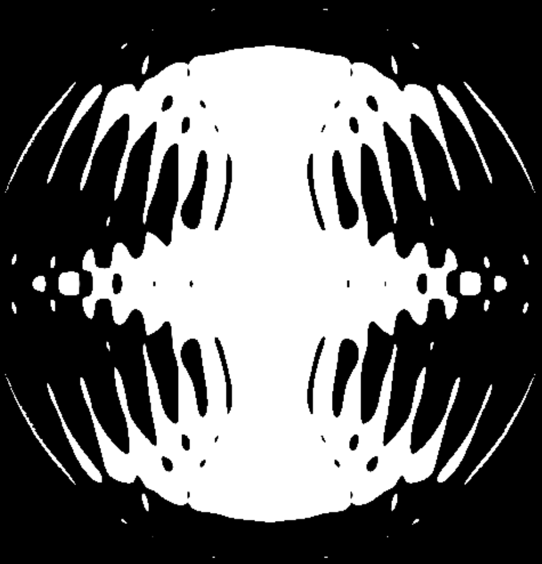


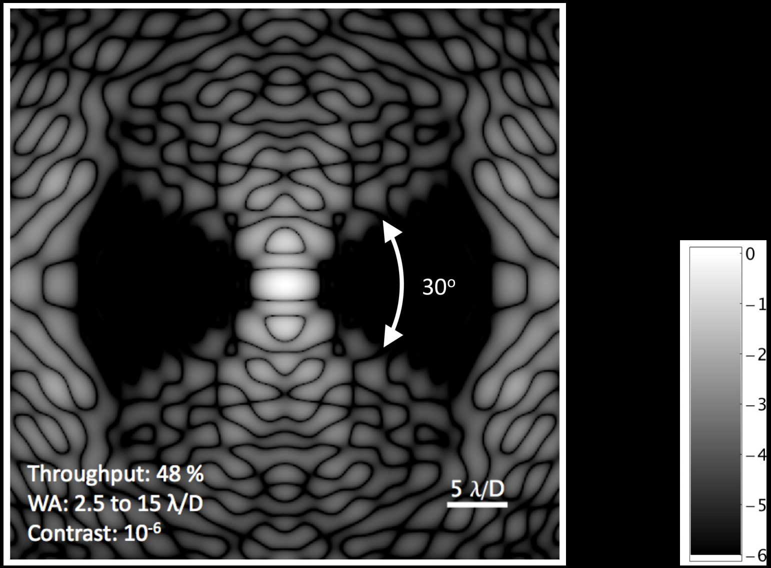

Throughput: 25 %

WA: 3.5 to 7 λ/D

Contrast: 10^{-10}

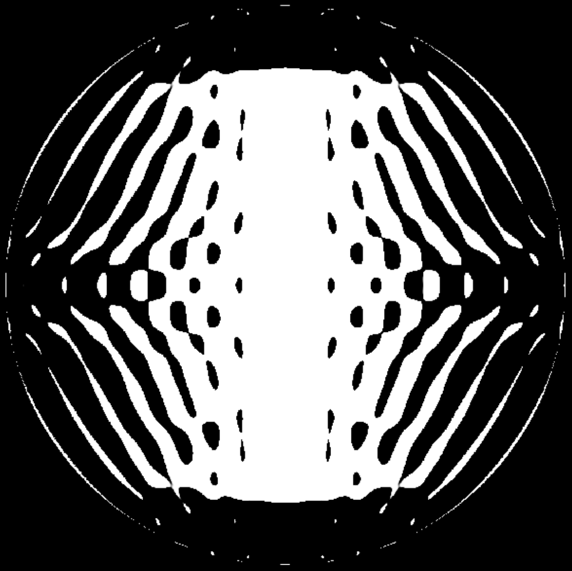
Circular Unobstructed Pupil

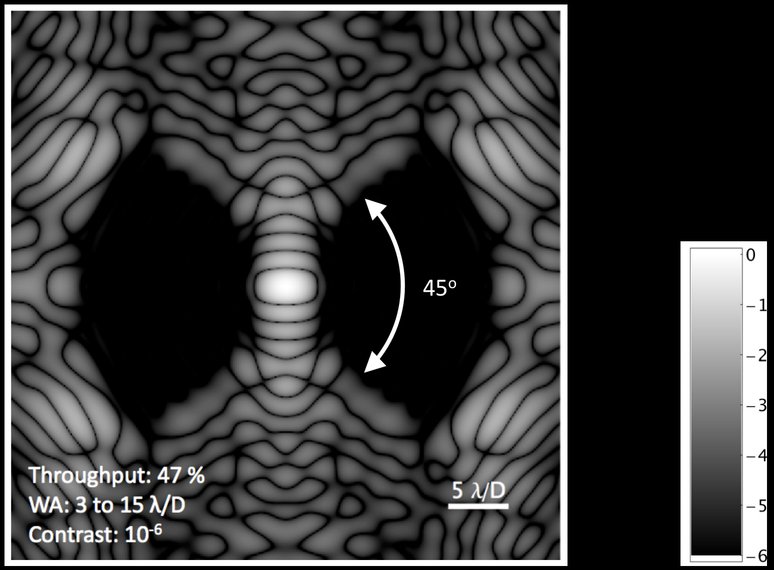




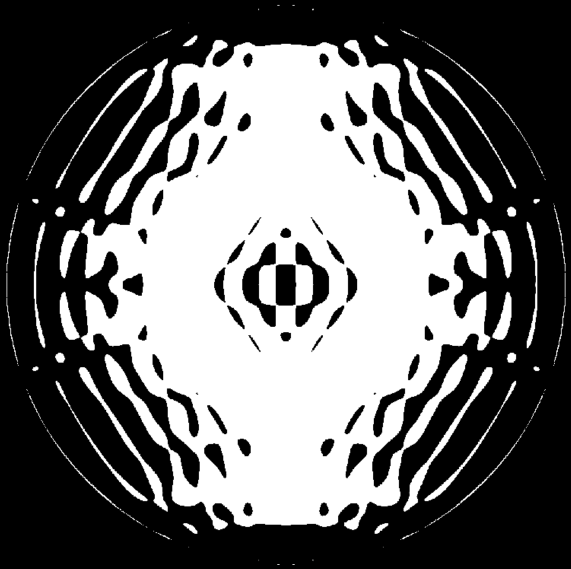
Throughput: 48 %
WA: 2.5 to 15 λ/D
Contrast: 10^{-6}

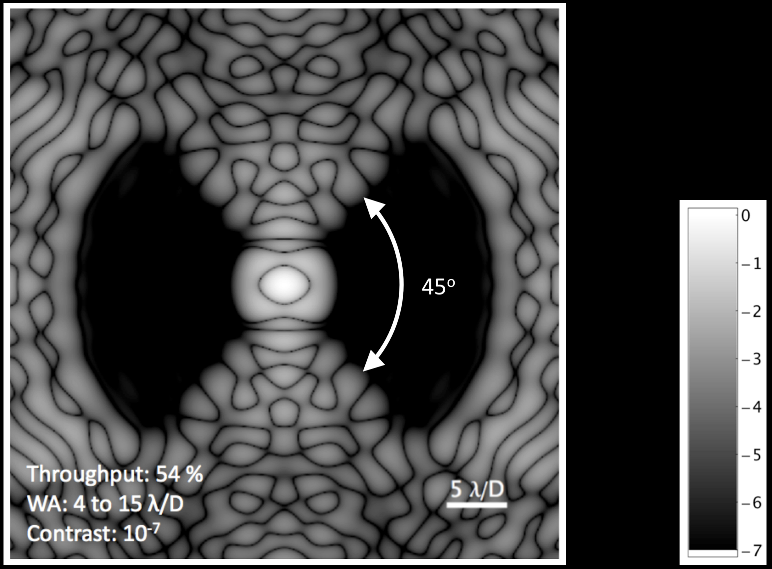
Circular Unobstructed Pupil





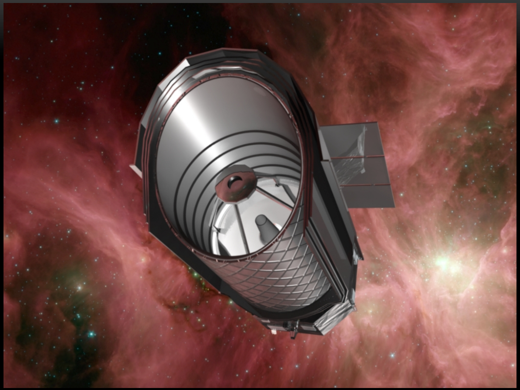
Circular Unobstructed Pupil

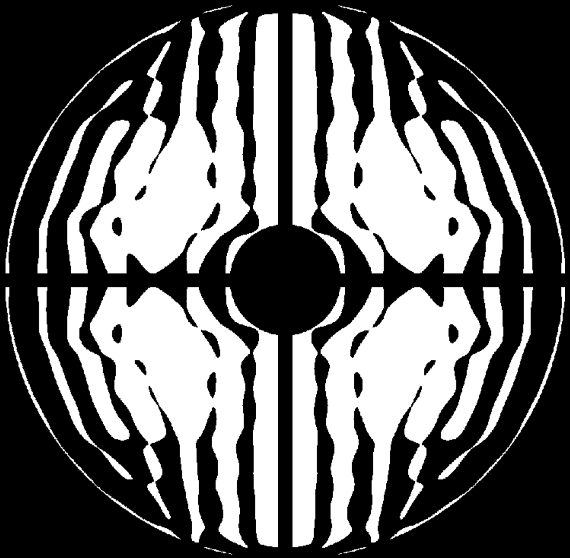


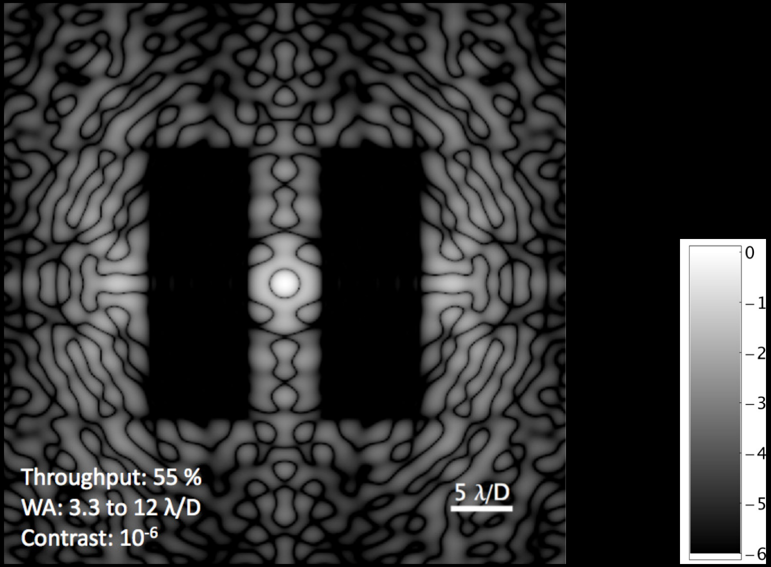


Throughput: 54 %
WA: 4 to 15 λ/D
Contrast: 10^{-7}

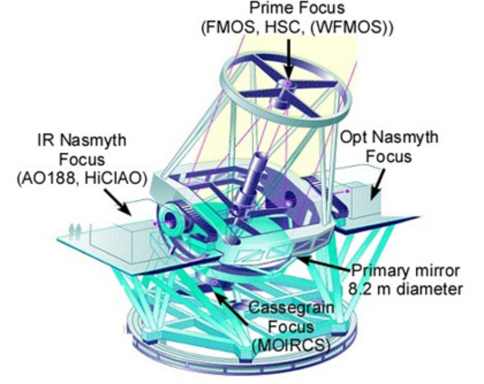
SPICA

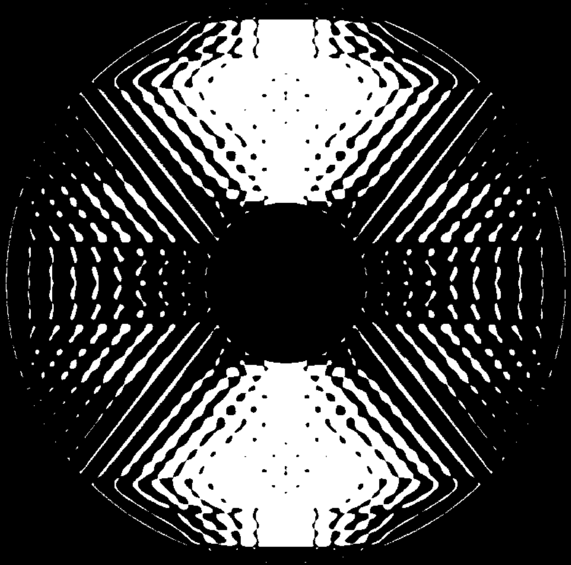


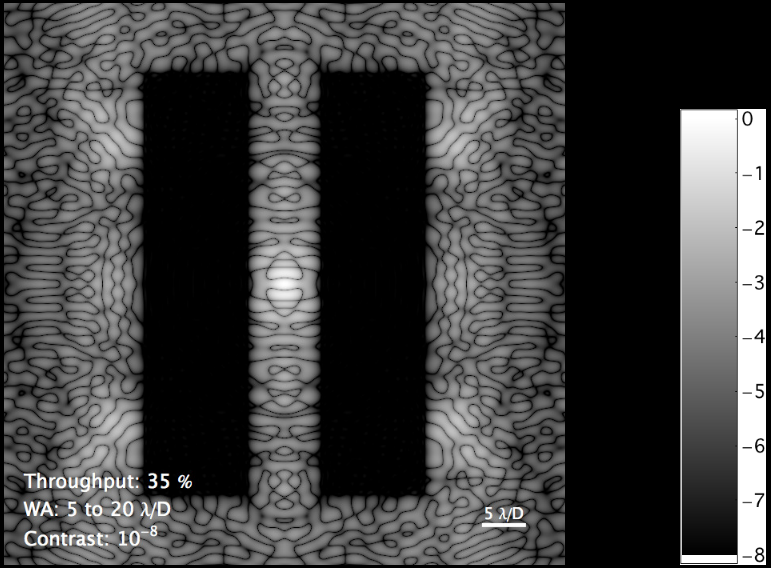




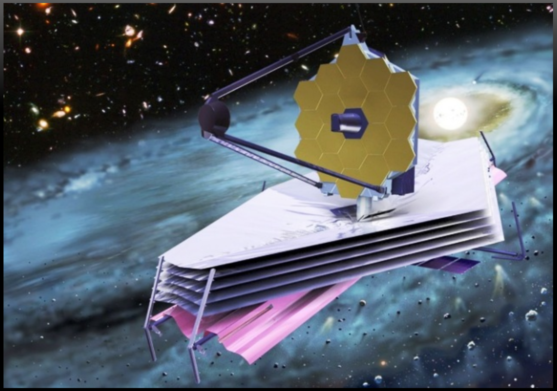
Subaru

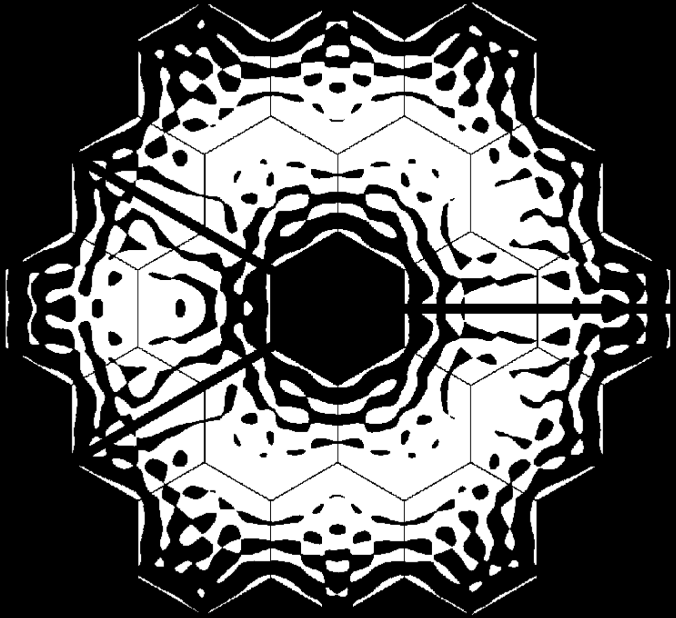

Prime Focus
(FMOS, HSC, (WFMOS))
IR Nasmyth
Focus
(AO188, HiCIAO)
Opt Nasmyth
Focus
Primary mirror
8.2 m diameter
Cassegrain
Focus
(MOIRCS)

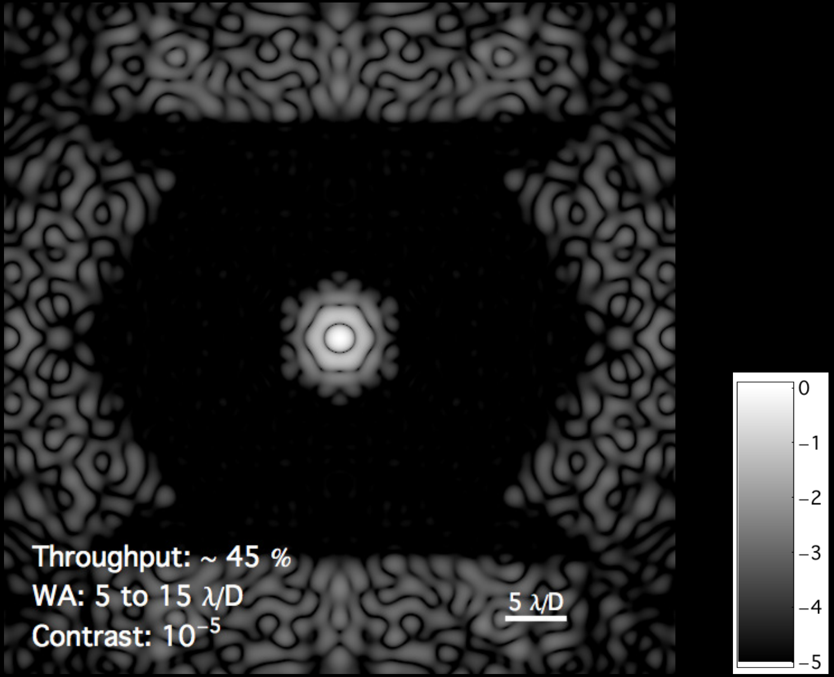



Throughput: 35 %
WA: 5 to 20 λ/D
Contrast: 10^{-8}
5 λ/D
0
−1
−2
−3
−4
−5
−6
−7
−8

JWST







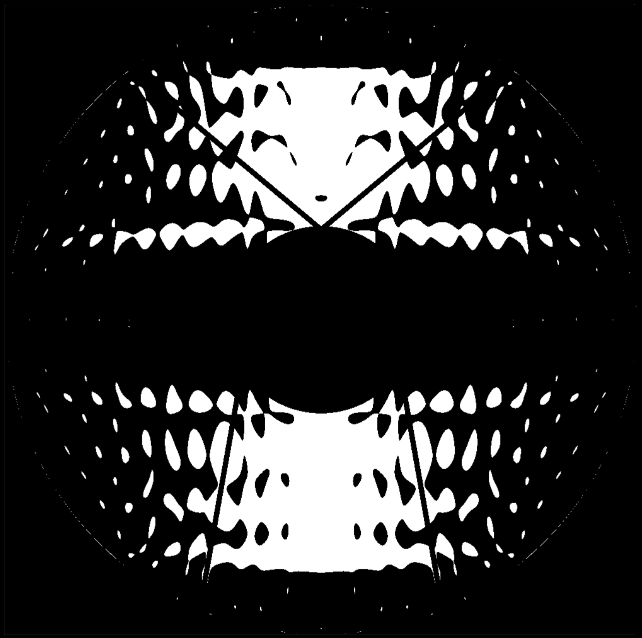
Throughput: ~ 45 %

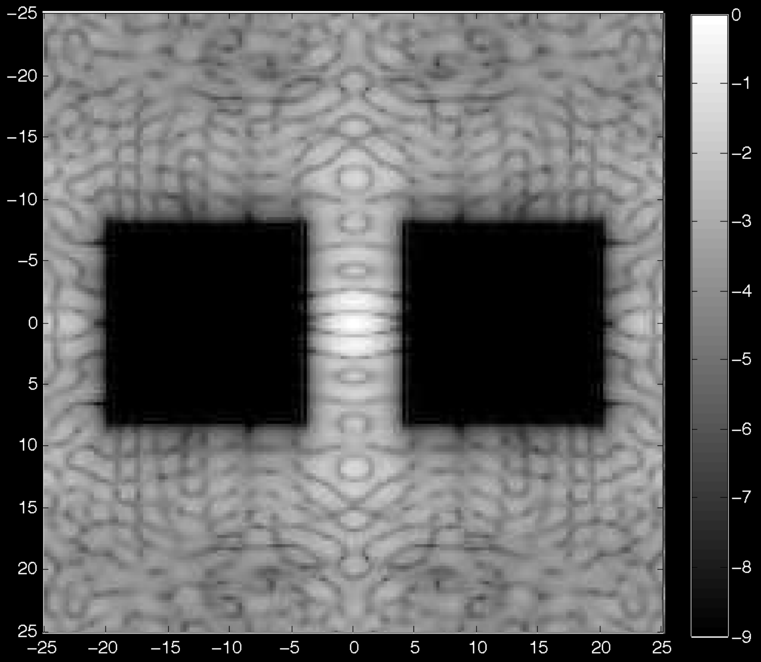
WA: 5 to 15 λ/D

Contrast: 10^{-5}

AFTA Telescope

- iwa $= 3.5\ \lambda/D$, owa $= 20\ \lambda/D$
- contrast $= 10^{-9}$
- throughput $\approx 10\%$
- central lobe $\approx 31\%$, trilobe $\approx 76\%$





Thank You!