A Simple Approximate Analysis of the Linear Stability of Ring Systems

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Abstract. We give a simple linear stability analysis of a system of n equal mass bodies in circular orbit about a single more massive body. A full analysis requires the possibility of perturbing all bodies. If the massive body is sufficiently dominant, then one can ignore perturbations to it. In this paper, we give a linear stability analysis based on perturbations to just one of the small ring bodies. Such an analysis could be justified by assuming that this one body has mass zero. But, we do not make this assumption. Therefore, it is surprising that the result we obtain agrees to within a factor of 2 with the result one obtains by considering perturbations to all ring bodies. We also give a simple back-of-the-envelope computation that shows that our stability mass threshold is consistent with the observed optical density of Saturn's rings.

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INTRODUCTION

One hundred and fifty years ago, Maxwell [1] was awarded the prestigious Adam's prize for a paper on the stability of Saturn's rings. At that time, neither the structure nor the composition of the rings was known. Hence, Maxwell considered various scenarios such as the possibility that the rings were solid or liquid annuli or a myriad of small boulders. As a key part of this last possibility, Maxwell studied the case of n equal-mass bodies orbiting Saturn at a common radius and uniformly distributed about a circle of this radius. He concluded that, for large n, the ring ought to be stable provided that the following inequality is satisfied:

$$\text{mass(Rings)} \leq 2.298\text{mass(Saturn)}/n^2.$$

The mathematical analysis that leads to this result has been scrutenized, validated, and generalized by a number of mathematicians over the years.

We summarize briefly some of the key historical developments. Tisserand [2] derived the same stability criterion using an analysis where he assumed that the ring has no effect on Saturn and that the highest vibration mode of the system controls stability. More recently, Willerding [3] used the theory of density waves to show that Maxwell's results are correct in the limit as n goes to infinity. Pendse [4] reformulated the stability problem so that it takes into account the effect of the rings on the central body. He proved that, for $n \leq 6$, the system is unconditionally unstable. Inspired by this work, Salo and Yoder [5] studied coorbital formations of n satellites for small values of n where the satellites are not distributed uniformly around the central body. They showed that there are some stable asymmetric formations (such as the well-known case of a pair of ring bodies in L4/L5 position relative to each other—i.e., one leading the other by 60 deg). Finally, Scheeres and Vinh [6] extended the analysis of Pendse to find the stability criterion as a function of the number of satellites when n is small. The resulting threshold depends on n but for $n \geq 7$, it deviates only a small amount from the asymptotically derived value. A full and self-contained analysis can be found in Vanderbei and Kolemen [7].

In this paper, we give a simplified linear stability analysis of a system of equal mass bodies in circular orbit about a single more massive body. The simplification stems from the fact that we consider only perturbations to one of the ring bodies.

Throughout the paper we consider only the planar n-body problem. That is, we ignore any instabilities that might arise due to out-of-plane perturbations. Maxwell claimed, and others have confirmed, that these out-of-plane perturbations are less destabilizing than in-plane ones and hence our analysis, while not fully three dimensional, does get close to the right answer.

Consider the multibody problem consisting of one large central body, say Saturn, having mass M and n small masses, such as boulders, each of mass m orbiting the large body in circular orbits uniformly spaced in a ring of radius r. Indices 0 to $n-1$ will be used to denote the ring masses and index n will be used for Saturn.

Using notations of the complex plane, we can write for $j = 0, 1, \ldots, n-1$,

$$z_j = re^{i(\omega t + 2\pi j/n)} \tag{1}$$

and

$$z_n = 0. \tag{2}$$

By symmetry, force is balanced on Saturn itself. For the ring bodies, again by symmetry, it suffices to consider just one of them, say body 0. Differentiating (1), we see that

$$\ddot{z}_0 = -\omega^2 z_0. \tag{3}$$

From Newton's law of gravity we have that

$$\ddot{z}_0 = -GM \frac{z_0}{|z_0|^3} + \sum_{k=1}^{n-1} Gm \frac{z_k - z_0}{|z_k - z_0|^3}. \tag{4}$$

Now, it is easy to check that

$$z_k - z_0 = re^{i\omega t} e^{\pi ik/n} 2i \sin(\pi k/n) \tag{5}$$

and hence that

$$|z_k - z_0| = 2r \sin(\pi k/n). \tag{6}$$

Substituting (5) and (6) into (4) and equating this with (3), we see that

$$\begin{aligned} -\omega^2 &= -\frac{GM}{r^3} + \sum_{k=1}^{n-1} \frac{Gm}{4r^3} \frac{ie^{\pi ik/n}}{\sin^2(\pi k/n)} && (7) \\ &= -\frac{GM}{r^3} - \frac{Gm}{4r^3} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)} + i \frac{Gm}{4r^3} \sum_{k=1}^{n-1} \frac{\cos(\pi k/n)}{\sin^2(\pi k/n)}. && (8) \end{aligned}$$

It is easy to check that the summation in the imaginary part on the right vanishes. Hence,

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{r^3} I_n \tag{9}$$

where

$$I_n = \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)}. \tag{10}$$

FIRST-ORDER STABILITY

In order to perform a stability analysis, we need to counter-rotate the system so that all bodies remain at rest. We then perturb the system slightly and analyze the result.

The counter-rotated system is given by

$$w_j = e^{-i\omega t} z_j = re^{2\pi ij/n} = u_j + iv_j. \tag{11}$$

Considering again just $j = 0$ and differentiating twice, we get

$$\ddot{w}_0 = e^{-i\omega t} \ddot{z}_0 - 2i\omega \dot{w}_0 + \omega^2 w_0. \tag{12}$$

Substituting (4) into (12), we see that

$$\ddot{w}_0 = -GM \frac{w_0}{|w_0|^3} + \sum_{k=1}^{n-1} Gm \frac{w_k - w_0}{|w_k - w_0|^3} - 2i\omega \dot{w}_0 + \omega^2 w_0. \tag{13}$$

Now, let

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \xi_4(t) \end{bmatrix} = \begin{bmatrix} u_0(t) \\ v_0(t) \\ \dot{u}_0(t) \\ \dot{v}_0(t) \end{bmatrix} \tag{14}$$

and consider the nonlinear system defined by

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} \xi_3 \\ \xi_4 \\ -GM\frac{\xi_1}{(\xi_1^2+\xi_2^2)^{3/2}} + \sum_k Gm \frac{u_k-\xi_1}{\left((u_k-\xi_1)^2+(v_k-\xi_2)^2\right)^{3/2}} + 2\omega\xi_4 + \omega^2\xi_1 \\ -GM\frac{\xi_2}{(\xi_1^2+\xi_2^2)^{3/2}} + \sum_k Gm \frac{v_k-\xi_2}{\left((u_k-\xi_1)^2+(v_k-\xi_2)^2\right)^{3/2}} - 2\omega\xi_3 + \omega^2\xi_2 \end{bmatrix}. \tag{15}$$

Choosing ω according to (9) makes

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \xi_4(t) \end{bmatrix} \equiv \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

a fixed point of this nonlinear system. Let

$$\begin{bmatrix} \delta\xi_1(t) \\ \delta\xi_2(t) \\ \delta\xi_3(t) \\ \delta\xi_4(t) \end{bmatrix} = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \xi_4(t) \end{bmatrix} - \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

denote a variation about this fixed point. Then, a first-order expansion of the nonlinear system yields

$$\begin{bmatrix} \delta\dot{\xi}_1 \\ \delta\dot{\xi}_2 \\ \delta\dot{\xi}_3 \\ \delta\dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_{31} & A_{32} & 0 & 2\omega \\ A_{41} & A_{42} & -2\omega & 0 \end{bmatrix} \begin{bmatrix} \delta\xi_1 \\ \delta\xi_2 \\ \delta\xi_3 \\ \delta\xi_4 \end{bmatrix} \tag{18}$$

where

$$A_{31} = -GM\frac{-2\xi_1^2+\xi_2^2}{(\xi_1^2+\xi_2^2)^{5/2}} + \sum_k Gm\frac{2(u_k-\xi_1)^2-(v_k-\xi_2)^2}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} + \omega^2 \tag{19}$$

$$A_{32} = 3GM\frac{\xi_1\xi_2}{(\xi_1^2+\xi_2^2)^{5/2}} + 3\sum_k Gm\frac{(u_k-\xi_1)(v_k-\xi_2)}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} \tag{20}$$

$$A_{41} = 3GM\frac{\xi_1\xi_2}{(\xi_1^2+\xi_2^2)^{5/2}} + 3\sum_k Gm\frac{(u_k-\xi_1)(v_k-\xi_2)}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} \tag{21}$$

$$A_{42} = -GM\frac{\xi_1^2-2\xi_2^2}{(\xi_1^2+\xi_2^2)^{5/2}} + \sum_k Gm\frac{-(u_k-\xi_1)^2+2(v_k-\xi_2)^2}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} + \omega^2 \tag{22}$$

From (11) we see that

$$u_k = r\cos(2\pi k/n) \text{ and } v_k = r\sin(2\pi k/n). \tag{23}$$

Hence,

$$u_k - u_0 = r\cos(2\pi k/n) - r = -2r\sin^2(\pi k/n) \tag{24}$$

and

$$v_k - v_0 = r\sin(2\pi k/n) = 2r\sin(\pi k/n)\cos(\pi k/n). \tag{25}$$

From these expressions, we get that

$$\sum_k \frac{(u_k-\xi_1)^2}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} = \frac{1}{(2r)^3}\sum_k \frac{1}{\sin(\pi k/n)} = \frac{1}{2r^3}I_n, \tag{26}$$

$$\sum_k \frac{(u_k-\xi_1)(v_k-\xi_2)}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} = \frac{1}{(2r)^3}\sum_k \frac{\cos(\pi k/n)}{\sin^2(\pi k/n)} = 0, \tag{27}$$

and

$$\sum_k \frac{(v_k-\xi_2)^2}{((u_k-\xi_1)^2+(v_k-\xi_2)^2)^{5/2}} = \frac{1}{(2r)^3}\sum_k \frac{\cos^2(\pi k/n)}{\sin^3(\pi k/n)} = \frac{1}{2r^3}(J_n - I_n), \tag{28}$$

where

$$J_n = \frac{1}{4}\sum_k \frac{1}{\sin^3(\pi k/n)}. \tag{29}$$

Substituting these expressions into (19)-(22), we get that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 - \frac{Gm}{2r^3}I_n - \frac{Gm}{2r^3}J_n & 0 & 0 & 2\omega \\ 0 & -\frac{Gm}{2r^3}I_n + \frac{Gm}{r^3}J_n & -2\omega & 0 \end{bmatrix}. \tag{30}$$

We are interested in large n. For all n, $0 \le I_n \le J_n$, and, for large n, J_n is orders of magnitude larger than I_n (the ratio grows quadratically). Hence, we drop the I_n terms and use the following approximation for A:

$$A \approx \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 - \frac{Gm}{2r^3}J_n & 0 & 0 & 2\omega \\ 0 & \frac{Gm}{r^3}J_n & -2\omega & 0 \end{bmatrix}. \tag{31}$$

It is easy to see that the characteristic polynomial for a matrix in the form

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & d \\ b & c & -d & 0 \end{bmatrix} \tag{32}$$

is

$$\lambda^4 + (d^2 - a - c)\lambda^2 + (ac - b^2). \tag{33}$$

By the quadratic formula, the eigenvalues of the matrix are the four values of λ defined by

$$\lambda^2 = \frac{1}{2}\left(a + c - d^2 \pm \sqrt{(a+c-d^2)^2 - 4(ac-b^2)}\right). \tag{34}$$

Applying this formula to the matrix at hand, we see that

$$\lambda^2 = \frac{1}{2}\left(-\omega^2 + \frac{Gm}{2r^3}J_n \pm \sqrt{\omega^4 - 13\omega^2\frac{Gm}{r^3}J_n - \frac{7}{4}\left(\frac{Gm}{r^3}J_n\right)^2}\right). \tag{35}$$

The solution to the differential equation is linearly stable if none of the eigenvalues have positive real part. This is equivalent to the following two inequalities:

$$\omega^4 - 13\omega^2\frac{Gm}{r^3}J_n - \frac{7}{4}\left(\frac{Gm}{r^3}J_n\right)^2 \ge 0 \tag{36}$$

and

$$\left(-\omega^2 + \frac{Gm}{2r^3}J_n\right)^2 \ge \omega^4 - 13\omega^2\frac{Gm}{r^3}J_n - \frac{7}{4}\left(\frac{Gm}{r^3}J_n\right)^2. \tag{37}$$

Let

$$t = \frac{\omega^2}{GmJ_n/r^3}.$$

Then, (36) and (37) are equivalent to

$$\left(-t+\frac{1}{2}\right)^2 \geq t^2 - 13t - \frac{7}{4} \geq 0. \tag{38}$$

The first inequality is equivalent to $t \geq -1/6$ and the second inequality is equivalent to

$$t \leq \frac{13-\sqrt{176}}{2} \text{ or } t \geq \frac{13+\sqrt{176}}{2}.$$

Since we are only interested in nonnegative values for t, these inequalities simplify to

$$t \geq \frac{13+\sqrt{176}}{2} = 13.13.$$

Summarizing, the second inequality can be written as

$$m \leq \frac{M}{13.13J_n}.$$

For large n, $J_n \approx 0.0775n^3$ (and $I_n \approx 0.642n\log(n)$). Hence we can write

$$\text{mass(Rings)} \leq 0.978\text{mass(Saturn)}/n^2.$$

Maxwell [1] studied this problem back in the middle of the 19th century and got a different answer:

$$\text{mass(Rings)} \leq 2.298\text{mass(Saturn)}/n^2.$$

This discrepancy is surprising. Furthermore, the analysis presented above is based on the simplifying assumption that only one body needs to be perturbed. If this body had been a massless body, then this simplifying assumption would be okay. But, in fact, we've assumed each ring body has the same mass. Hence, when we perturb a body slightly, that perturbation induces a gravitational effect on nearby bodies. They then are perturbed and that perturbation feeds back to the original perturbed ring body. This feedback loop tends to destabilize the system. Hence, the above analysis provides what should be an overly optimistic estimate for the upper bound on stability. In other words, the factor 0.978 is too high. In fact, careful numerical simulation shows that the stability threshold factor is closer to 0.5.

Suppose that the linear density of the boulders is λ. That is, λ is the ratio of the diameter of one boulder to the separation between the centers of two adjacent boulders. Then the diameter of a single boulder is $\lambda(2\pi r/n)$. Hence, the volume of a single boulder is $(4\pi/3)(\lambda\pi r/n)^3$. Let δ denote the density of a boulder. Then the mass of a single boulder is $(4\pi/3)(\lambda\pi r/n)^3\delta$. If we assume that the density of Earth is 7.9 times that of a boulder (Earth's density is 5.5 and Saturn's moons have a density of about 0.7 being composed of porous water-ice), then we have

$$\delta = \frac{1}{7.9}\frac{m_E}{(4\pi/3)r_E^3},$$

where m_E denotes the mass of Earth and r_E denotes its radius. Combining all of these factors and assuming the central mass is equal to Saturn's mass and the ring's radius is about the radius of the Cassini division ($120,000$km), we see that the upper bound on the linear density of boulders is

$$\lambda \leq \left(7.9\frac{M/m_E}{(13.13)(0.0775)}\right)^{1/3}\frac{r_E}{r} = 0.155.$$

In other words, the linear density cannot exceed 15% otherwise the ring will be unstable. Of course, this is for a one dimensional circular ring of ice boulders. Analysis of a two dimensional annulus or the full three dimensional case is naturally more complicated. Nonetheless the 15% linear density figure matches surprisingly well with the measured optical density which hovers around 0.05 to 2.5.

Final Notes

Let us return to the exact expression for A as given by (30). If we do not drop the I_n terms from A, the subsequent methods remain the same although the details become more tedious. The result is that the system is linearly stable when

$$\omega^2 \geq \frac{1}{2}\left(13J_n - 7I_n + 2J_n\sqrt{(2-I_n/J_n)(22-13I_n/J_n)}\right).$$

Finally, (9) then yields the following inequality:

$$\frac{M}{m} \geq \frac{1}{2}\left(13J_n - 9I_n + 2J_n\sqrt{(2-I_n/J_n)(22-13I_n/J_n)}\right).$$

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