

# THE COMPLEX ZEROS OF RANDOM SUMS

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*This paper is dedicated to the memory of Larry Shepp.*

Random sums, be they polynomials, Fourier series, or sums of other basis functions, arise frequently in applied mathematics. Understanding their characteristics in a probabilistic sense is both interesting and practical. In particular, a great deal of work has been devoted to understanding the distribution of the roots of polynomials whose coefficients are independent, identically distributed Normal random variables. Early work on such random polynomials focused on determining the expected number of real roots. Later contributions include (i) deriving an explicit formula for the density of the roots in the complex plane, (ii) quantifying the fact that the complex roots cluster near the unit circle, and (iii) extensions to the case where the distribution is not Gaussian. In this paper, we describe a generalization of the formula for the explicit density to the case where the terms of the random sum are not restricted to be powers of the complex variable  $z$ . In particular, we give an explicit formula for the complex roots of random Fourier sums.

## 1. Introduction. Let

$$P_n(z) = \sum_{j=0}^n \eta_j f_j(z), \quad z \in \mathbf{C},$$

where  $n$  is a fixed integer, the  $\eta_j$ 's are independent identically distributed  $N(0,1)$  random variables, and the functions  $f_j$  are given entire functions that are real-valued on the real line. We derive an explicit formula for the expected number of zeros in any measurable subset  $\Omega$  of the complex plane

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C. The formula will be expressed in terms of the following functions:

$$\begin{aligned} A_0(z) &= \sum_{j=0}^n f_j(z)^2, & B_0(z) &= \sum_{j=0}^n |f_j(z)|^2, \\ A_1(z) &= \sum_{j=0}^n f_j(z)f'_j(z), & B_1(z) &= \sum_{j=0}^n \overline{f_j(z)}f'_j(z), \\ A_2(z) &= \sum_{j=0}^n f'_j(z)^2, & B_2(z) &= \sum_{j=0}^n |f'_j(z)|^2, \end{aligned}$$

$$(1) \quad D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2},$$

and

$$E_1(z) = \sqrt{A_2(z)A_0(z) - A_1(z)^2}.$$

Note that we follow the usual notational conventions: an overbar denotes complex conjugation, primes denote differentiation with respect to  $z$  and the real and imaginary parts of a complex variable  $z$  are denoted by  $x$  and  $y$ , respectively, i.e.,  $z = x + iy$ . Also, we should mention right from the start that the function  $E_1$  will only be needed in places where the argument of the square root is a positive real. At such places, the square root is assumed to be the positive real root.

Let  $\nu_n(\Omega)$  denote the (random) number of zeros of  $P_n$  in a set  $\Omega$  in the complex plane. Our first theorem asserts that over most of the plane this random variable can be characterized by a density with respect to Lebesgue measure on the plane:

**THEOREM 1.1.** *For each measurable set  $\Omega \subset \{z \in \mathbf{C} \mid D_0(z) \neq 0\}$ ,*

$$(2) \quad \mathbf{E}\nu_n(\Omega) = \int_{\Omega} h_n(x, y) dx dy,$$

where

$$h_n(z) = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0} \overline{B_1} A_1)}{\pi D_0^3}.$$

It is easy to see that the density function  $h_n$  is real valued. The fact that it is nonnegative can be taken as a corollary of this theorem. A simple direct proof is not obvious. As we see from the above theorem, places where  $D_0$  vanishes are special and must be studied separately. The real axis is one such place:

**THEOREM 1.2.** *On the real line, the function  $D_0$  vanishes. For each Borel measurable set  $\Omega \subset \mathbb{R}$ ,*

$$(3) \quad \mathbf{E}\nu_n(\Omega) = \int_{\Omega} g_n(x) dx,$$

where

$$g_n(x) = \frac{E_1(x)}{\pi B_0(x)}.$$

In the case where the  $f_j(z)$ 's are just powers of  $z$ , our results reduce to those given in [22]. The proof in Section 5 parallels the analogous proof given in [22] but there are a few important differences, the most significant one being the fact that, in general, the function  $B_1$  is not real-valued like it was in [22]. This explicit formula has not been derived before and, as shown in the following section, there are interesting new examples that can now be solved.

While the definition of  $h_n$  in Theorem 1.1 looks rather complicated, it is nevertheless amenable both to computation and, with some choices of the functions  $f_j$ , asymptotic analysis as well.

In the following section, we look at some specific examples. Then, in Section 2, we derive the explicit formulas given above for the intensity functions  $h_n$  and  $g_n$ .

**2. Examples.** In this section, we consider some examples. Most figures in this section show two plots. On the left is a grey-scale plot of the intensity functions  $g_n$  and  $h_n$ . On the right is a plot of thousands of zeros obtained by generating random sums and numerically finding their zeros.

*2.1. Polynomials with iid Gaussian Coefficients.* The standard example corresponds to the  $f_j$  simply being the power functions:

$$f_j(z) = z^j.$$

As this case was studied carefully in [22], other than showing a particular example ( $n = 10$ ) in Figure 1, we refer the reader to that previous paper for more information about this example.

The intensity plots appearing on the left were produced by partitioning the given square domain into a 440 by 440 grid of “pixels” and computing the intensity function in the center of each pixel. The grey-scale was computed by assigning black to the pixel with the smallest value and white to the pixel with the largest value and then linearly interpolating all values in between. This grey-scale computation was performed separately for  $h_n$  and for  $g_n$

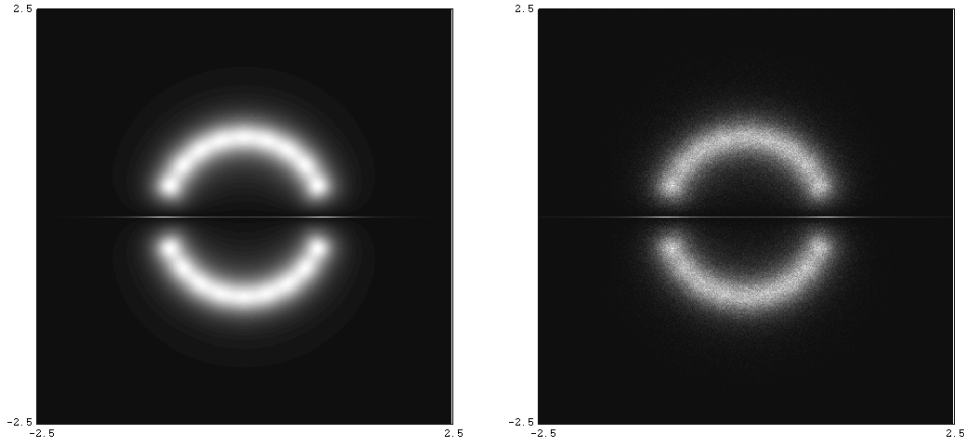


FIG 1. *Random degree 10 polynomial:  $\eta_0 + \eta_1 z + \eta_2 z^2 + \dots + \eta_{10} z^{10}$ . In this figure and the following ones, the left-hand plot is a grey-scale image of the intensity functions  $h_n$  and  $g_n$  (the latter being concentrated on the x-axis). The right-hand plot shows 200,000 roots from randomly generated polynomials. Note that, for the left-hand plots, the grey-scales for  $h_n$  and  $g_n$  are scaled separately and in such a way that both use the full range from white to black.*

(which appears only on the x-axis) and so no conclusions should be drawn comparing the intensity shown on the x-axis with that shown off from it. The Javascript applet used to produce this and the other figures in this section can be found at

<https://vanderbei.princeton.edu/WebGL/roots.html>

Of course, the intensity function  $g_n$  is one-dimensional and therefore it would be natural (and more informative) to make separate plots of values of  $g_n$  as a function of the real variable  $x$ . For this example, such a plot appears in many places (see, e.g., [16]) and so it seems unnecessary to produce it here. We will, however, show such a plot for some of the following examples.

2.2. *Weyl Polynomials.* Sums in which the  $f_j$ 's are given by

$$f_j(z) = \frac{z^j}{\sqrt{j!}}$$

are called *Weyl polynomials* (also sometimes called *flat polynomials*). Figure 2 shows the empirical distribution for the case where  $n = 80$ . For this case, the limiting forms of the various functions defining the densities are easy to

compute:

$$\begin{aligned} \lim_{n \rightarrow \infty} A_0(z) &= e^{z^2} & \lim_{n \rightarrow \infty} B_0(z) &= e^{|z|^2} \\ \lim_{n \rightarrow \infty} A_1(z) &= ze^{z^2} & \lim_{n \rightarrow \infty} B_1(z) &= \bar{z}e^{|z|^2} \\ \lim_{n \rightarrow \infty} A_2(z) &= (z^2 + 1)e^{z^2} & \lim_{n \rightarrow \infty} B_2(z) &= (|z|^2 + 1)e^{|z|^2} \\ \lim_{n \rightarrow \infty} D_0(z) &= \sqrt{e^{2|z|^2} - e^{(z^2 - \bar{z}^2)}} & \lim_{n \rightarrow \infty} E_1(z) &= e^{z^2}. \end{aligned}$$

The random Weyl polynomials are interesting because in the limit as  $n \rightarrow \infty$ , the distribution of the non-real roots becomes independent of  $x$  and the distribution of the real roots becomes uniform over the real line:

**THEOREM 2.1.** *If  $f_j(z) = z^j / \sqrt{j!}$  for all  $j$ , then*

$$\lim_{n \rightarrow \infty} h_n(x, y) = \frac{1}{\pi} \cdot \frac{1 - (1 + 4y^2)e^{-4y^2}}{(1 - e^{-4y^2})^{3/2}}.$$

and

$$\lim_{n \rightarrow \infty} g_n(x) = \frac{1}{\pi}.$$

**PROOF.** Follows easily from Theorems 1.1 and 1.2 and the formulas above.  $\square$

In addition to the asymptotic uniformity of the distribution of the real roots, the density of the complex roots is almost exactly  $1/\pi$  everywhere except points close to the  $x$ -axis. In particular,

$$\lim_{|y| \rightarrow \infty} \lim_{n \rightarrow \infty} h_n(x, y) = \frac{1}{\pi}$$

and

$$\frac{0.9999982557}{\pi} \leq \lim_{n \rightarrow \infty} h_n(x, y) \leq \frac{1}{\pi}, \quad \text{for } |y| \geq 2$$

(see Figure 3).

Figure 4 shows plots of  $g_n$  for  $n = 10$  for the Weyl polynomial and for each of the other examples in this section.

**2.3. Taylor Polynomials.** Another obvious set of polynomials to consider are the random Taylor polynomials; i.e., those polynomials with

$$f_j(z) = \frac{z^j}{j!}.$$

Figure 5 shows the  $n = 10$  empirical distribution for these polynomials.

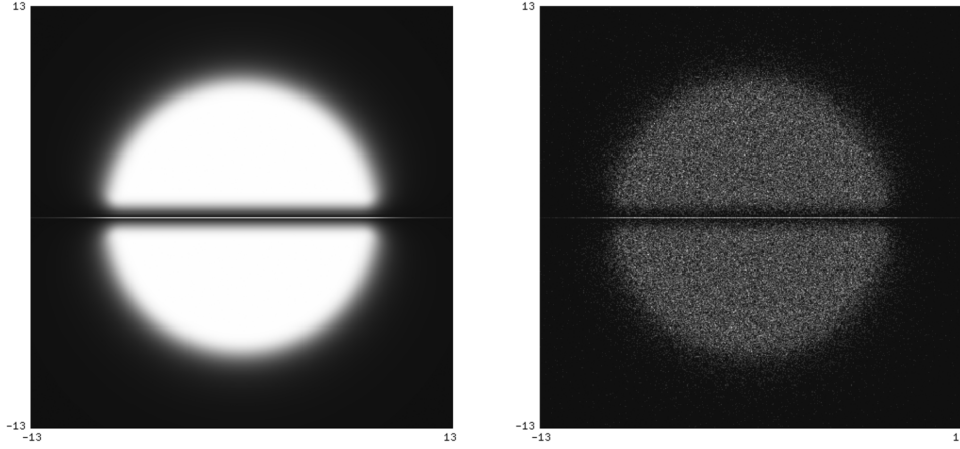


FIG 2. *Random degree 80 Weyl polynomials:  $\eta_0 + \eta_1 z + \eta_2 \frac{z^2}{\sqrt{2!}} + \cdots + \eta_{80} \frac{z^{80}}{\sqrt{80!}}$ . The empirical distribution on the right was generated using 2,500 random sums.*

2.4. *Root-Binomial Polynomials.* Let

$$f_j(z) = \sqrt{\binom{n}{j}} z^j.$$

Figure 6 shows the  $n = 10$  empirical distribution for these polynomials. This example is interesting because the real and complex density functions each take on a rather simple explicit form. Indeed, it is easy to check that

$$\begin{aligned} A_0(z) &= (1 + z^2)^n, & B_0(z) &= (1 + |z|^2)^n, \\ A_1(z) &= nz(1 + z^2)^{n-1}, & B_1(z) &= n\bar{z}(1 + |z|^2)^{n-1}, \\ A_2(z) &= n(1 + nz^2)(1 + z^2)^{n-2}, & B_2(z) &= n(1 + n|z|^2)(1 + |z|^2)^{n-2}. \end{aligned}$$

The formula for the density on the real axis simplifies nicely:

$$g(x) = \frac{\sqrt{n}}{\pi} \frac{1}{1 + x^2}.$$

From this formula, we see that the expected number of real roots is  $\sqrt{n}$  and that each real root has a Cauchy distribution.

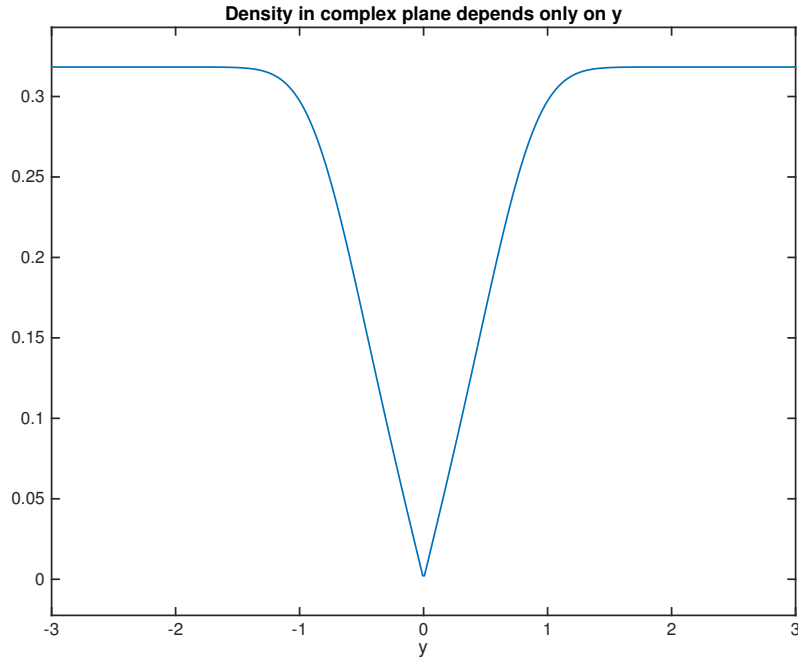


FIG 3. For the Weyl functions, the function  $\lim_{n \rightarrow \infty} h_n(x, y)$  depends only on  $y$  and is almost a constant for  $|y| \geq 2$ .

2.5. *Fourier Cosine Series.* Now let's consider a family of random sums that are not polynomials, namely, random (truncated) Fourier cosine series:

$$f_j(z) = \cos(jz).$$

This case is interesting because these functions are real-valued not only on the real axis but on the imaginary axis as well:  $\cos(iy) = \cosh(y)$ . Hence,  $D_0$  vanishes on both the real and the imaginary axes and, therefore, both axes have a density of zeros. The set of imaginary roots for a particular sum using the  $f_j$ 's map to a set of real roots if  $f_j(z)$  is replaced with  $\tilde{f}_j(z) = f_j(iz) = \cosh(z)$ . Hence, the formula for the density on the imaginary axis is easy to compute by this simple rotation. The resulting density on the imaginary axis has this simple form:

$$g_n(y) = \frac{E_1(iy)}{\pi B_0(iy)}.$$

An example with  $n = 10$  is shown in Figure 7.

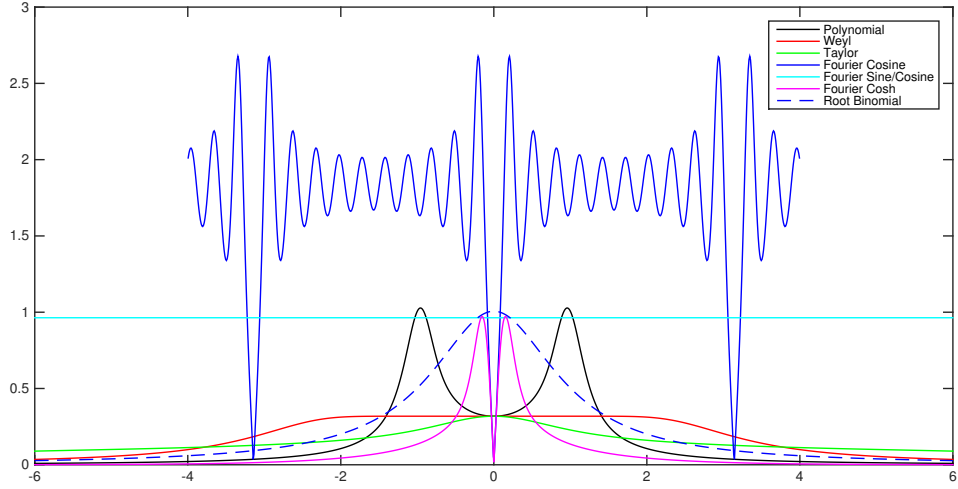


FIG 4. The function  $g_n$  for  $n = 10$  for several choices of the  $f_j$ 's

2.6. *Fourier Sine/Cosine Series.* Finally, we consider random (truncated) Fourier sine/cosine series:

$$f_j(z) = \begin{cases} \cos(\frac{j}{2}z), & j \text{ even,} \\ \sin(\frac{j+1}{2}z), & j \text{ odd.} \end{cases}$$

The  $n = 2$  case is shown in Figure 8 and the  $n = 10$  case is shown in Figure 9. For this example, it is easy to compute the key functions. Assuming that  $n$  is even and letting  $m = n/2$ , we get

$$\begin{aligned} A_0(z) &= m + 1 & B_0(z) &= m + 1 + 2 \sum_{j=1}^m \sinh^2(jy) \\ A_1(z) &= 0 & B_1(z) &= -2i \sum_{j=1}^m j \cosh(jy) \sinh(jy) \\ A_2(z) &= m(m+1)(2m+1)/6 & B_2(z) &= 2 \sum_{j=1}^m j^2 \sinh^2(jy) \end{aligned}$$

From these explicit formulas, it is easy to check that the density function  $h_n(z)$  depends only on the imaginary part of  $z$  as is evident in Figures 8 and 9. It is also easy to check that the distribution on the real axis is uniform;

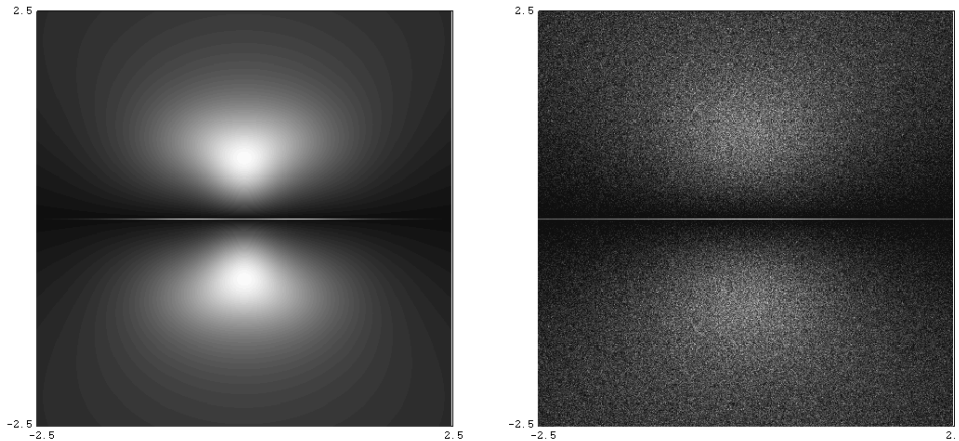


FIG 5. *Random degree 10 Taylor polynomials:  $\eta_0 + \eta_1 z + \eta_2 \frac{z^2}{2!} + \dots + \eta_{10} \frac{z^{10}}{10!}$ . The empirical distribution on the right was generated using 500,000 random sums.*

i.e., the density function  $g_n(x)$  is a constant:

$$g_n(x) = \frac{1}{2\pi} \sqrt{n(n+1)/3}.$$

We will prove our two main theorems in Section 5 but first, in the next section, we summarize some of the related work on this problem.

**3. Related Work.** The problem of characterizing the distribution of the roots of random polynomials has a long history. In 1943, Kac [15] studied the real roots of random polynomials with iid normal coefficients. He obtained an explicit formula for the density function for the distribution of the real roots.

Following the initial work of Kac, a large body of research on zeros of random polynomials has appeared – see [2] for a fairly complete account of the early work in this area including an extensive list of references. Most of this early work focused on the real zeros; [5], [8] and [26] being a few notable exceptions. The paper of Edelman and Kostlan [4] gives a very elegant geometric treatment of the problem.

In more recent years, the work has branched off in a number of directions. For example, in 1995, Larry Shepp and I [22] derived an explicit formula for the distribution of the roots in the complex plane when the coefficients are assumed to be iid normal random variables. A short time later, Ibragimov and Zeitouni [11] took a different approach and were able to rederive our

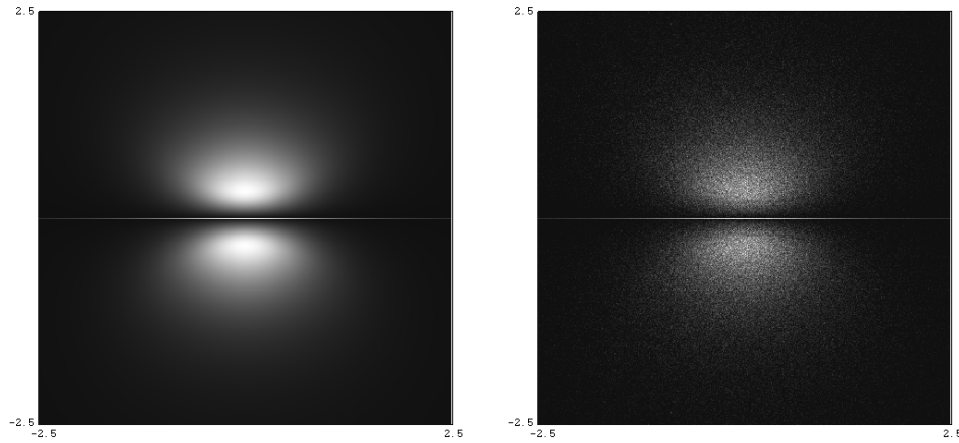


FIG 6. *Random degree 10 root-binomial polynomials:  $\eta_0 + \eta_1 \sqrt{\frac{10}{1}}z + \eta_2 \sqrt{\frac{10 \cdot 9}{2 \cdot 1}}z^2 + \eta_3 \sqrt{\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}}z^3 + \dots + \eta_{10}z^{10}$ . The empirical distribution on the right was generated using 50,000 random sums.*

results and also find limiting distributions as the degree  $n$  tends to infinity under more general distributional assumptions. See also [13] and [14].

Also in the late 1990's, it was pointed out that understanding deeper statistical properties of the random roots, such as  $k$ -point correlations among the roots, was both interesting mathematically and had important implications in physics (see, e.g., [20], [7] and [21]).

A number of papers have appeared that attempt to prove certain specific properties under increasingly general distributional assumptions. For example, in 2002, Dembo et al. [3] derived a formula for the probability that none of the roots are real (assuming  $n$  is even, of course) in the case when the coefficients of the polynomial are iid but not necessarily normal. Other papers have continued to study real roots—see, e.g., [12]. Another property that has been actively studied in recent years is the fact that as  $n$  gets large the complex roots tend to distribute themselves close to and uniformly about the unit circle in the complex plane—see, e.g., the papers by Shiffman and Zelditch [23], Hughes and Nikeghbali [9], Ibragimov and Zaporozhets [10], Pritsker and Yeager [19] and Pritsker [18]. Also, Li and Wei [17] have considered harmonic polynomials—polynomials in the complex variable  $z$  and its conjugate  $\bar{z}$ .

Using a very different approach, Feldheim [6] has derived a result that is equivalent to the results presented herein. However, her formulas are not as explicit those given here and deriving the formulas presented here starting

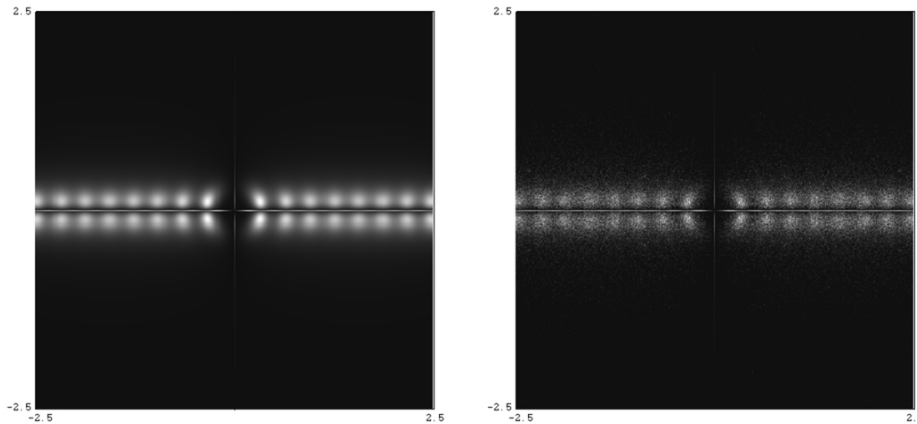


FIG 7. *Random sum of first 11 terms in a cosine Fourier series:  $\eta_0 + \eta_1 \cos(z) + \eta_2 \cos(2z) + \dots + \eta_{10} \cos(10z)$ . The empirical distribution on the right was generated using 24,000 random sums.*

from her result is not easy.

Recently, Tao and Vu [24], drawing on the close connection with random matrix theory, derived asymptotic formulas for the correlation functions of the roots of random polynomials. They specifically address the question of how many zeros are real.

The results summarized above mostly establish certain properties of the roots under very general distributional assumptions. The price paid for that generality is that most results only hold asymptotically as  $n \rightarrow \infty$ . In contrast, this paper introduces a modest generalization to the core assumptions underlying the results in [22] and we show that analogous explicit formulas can still be derived for any value of  $n$ .

An earlier version of this paper was posted on ArXiv [25]. That version ended with the comment that, if the coefficients are assumed to be independent complex Gaussians (instead of real), then the same methods can be applied and one would expect that the computations will be simpler. In this case, the intensity function does not have mass concentrated on the real axis (i.e.,  $g_n = 0$ ) and the intensity function is rotationally invariant. This extension has been carried out by Yeager [27].

We are now ready to prove our two main theorems.

**4. The Intensity Functions  $h_n$  and  $g_n$ .** This section is devoted to the proof of Theorems 1.1 and 1.2. We begin with the following proposition.

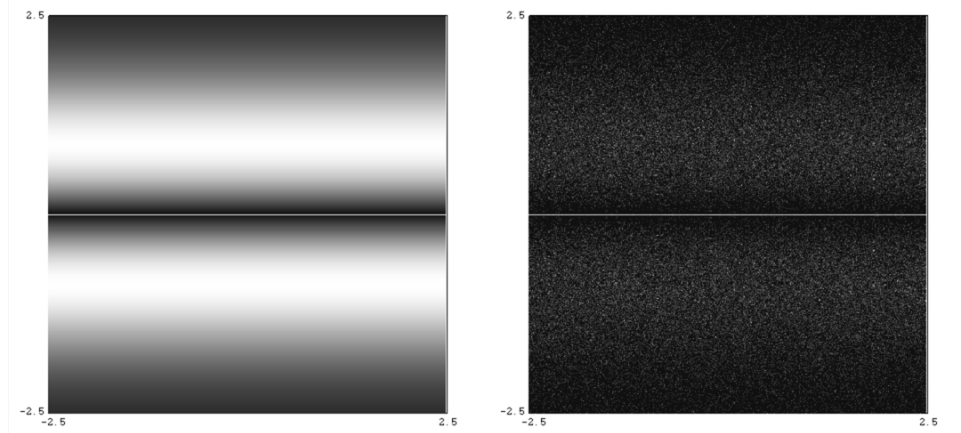


FIG 8. *Random sum of first 3 terms in a sine/cosine Fourier series:  $\eta_0 + \eta_1 \sin(z) + \eta_2 \cos(z)$ . The empirical distribution on the right was generated using 210,000 random sums.*

PROPOSITION 4.1. *For each set  $\Omega \subset \mathbf{C}$  whose boundary intersects the set  $\{z \mid D_0(z) = 0\}$  at most only finitely many times,*

$$(4) \quad \mathbf{E}\nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} F(z) dz,$$

where

$$(5) \quad F = \frac{B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1}{B_0 D_0 + B_0^2 - \bar{A}_0 A_0}.$$

PROOF. The argument principle (see, e.g., [1], p. 151) gives an explicit formula for the random variable  $\nu_n(\Omega)$ , namely

$$(6) \quad \nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{P'_n(z)}{P_n(z)} dz.$$

Taking expectations in (25) and then interchanging expectation and contour integration (the justification of which is tedious but doable), we get

$$(7) \quad \mathbf{E}\nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \mathbf{E} \frac{P'_n(z)}{P_n(z)} dz.$$

The following Lemma shows that, away from the set  $\{z \mid D_0(z) = 0\}$ , the function

$$(8) \quad F(z) = \mathbf{E} \frac{P'_n(z)}{P_n(z)}$$

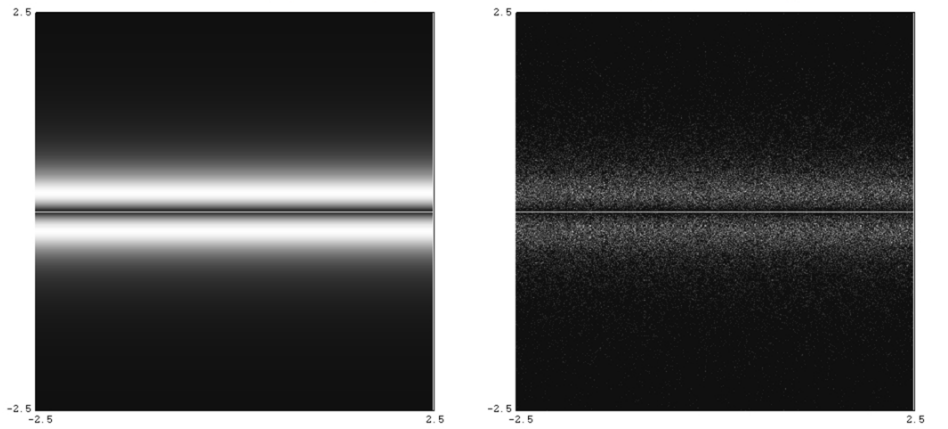


FIG 9. *Random sum of first 11 terms in a sine/cosine Fourier series:  $\eta_0 + \eta_1 \sin(z) + \eta_2 \cos(z) + \dots + \eta_9 \sin(5z) + \eta_{10} \cos(5z)$ . The empirical distribution on the right was generated using 26,000 random sums.*

simplifies to the expression given in (24) and, since we've assumed that  $\partial\Omega$  intersects this set at only finitely many points, this finishes the proof.  $\square$

LEMMA 4.2. *Let  $F$  denote the function defined by (27). For  $z \notin \{z \mid D_0(z) = 0\}$ ,*

$$F = \frac{B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1}{B_0 D_0 + B_0^2 - \bar{A}_0 A_0}.$$

PROOF. Note that  $P_n(z)$  and  $P'_n(z)$  are complex Gaussian random variables. It is convenient to work with their real and imaginary parts,

$$\begin{aligned} P_n(z) &= \xi_1 + i\xi_2, \\ P'_n(z) &= \xi_3 + i\xi_4, \end{aligned}$$

which are just linear combinations of the original standard normal random variables:

$$\begin{aligned} \xi_1 &= \sum_{j=0}^n a_j \eta_j, & \xi_2 &= \sum_{j=0}^n b_j \eta_j, \\ \xi_3 &= \sum_{j=0}^n c_j \eta_j, & \xi_4 &= \sum_{j=0}^n d_j \eta_j. \end{aligned}$$

The coefficients in these linear combinations are given by

$$(9) \quad \begin{aligned} a_j &= \operatorname{Re}(f_j(z)) = \frac{f_j(z) + \overline{f_j(z)}}{2}, \\ b_j &= \operatorname{Im}(f_j(z)) = \frac{f_j(z) - \overline{f_j(z)}}{2i}, \\ c_j &= \operatorname{Re}(f'_j(z)) = \frac{f'_j(z) + \overline{f'_j(z)}}{2}, \\ d_j &= \operatorname{Im}(f'_j(z)) = \frac{f'_j(z) - \overline{f'_j(z)}}{2i}. \end{aligned}$$

Put  $\xi = [\xi_1 \ \xi_2 \ \xi_3 \ \xi_4]^T$ . The covariance among these four Gaussian random variables is easy to compute:

$$(10) \quad \operatorname{Cov}(\xi) = \mathbf{E}\xi\xi^T = \begin{bmatrix} a^T a & a^T b & a^T c & a^T d \\ b^T a & b^T b & b^T c & b^T d \\ c^T a & c^T b & c^T c & c^T d \\ d^T a & d^T b & d^T c & d^T d \end{bmatrix}$$

We now represent these four correlated Gaussian random variables in terms of four independent standard normals. To this end, we seek a lower triangular matrix  $L = [l_{ij}]$  such that the vector  $\xi$  is equal in distribution to  $L\zeta$ , where  $\zeta = [\zeta_1 \ \zeta_2 \ \zeta_3 \ \zeta_4]^T$  is a vector of four independent standard normal random variables. The following simple calculation shows that  $L$  is the Cholesky factor for the covariance matrix:

$$(11) \quad \operatorname{Cov}(\xi) = \mathbf{E}\xi\xi^T = \mathbf{E}L\zeta\zeta^T L^T = LL^T.$$

Now, since  $\xi \stackrel{\text{D}}{=} L\zeta$  and  $L$  is lower triangular (the symbol  $\stackrel{\text{D}}{=}$  denotes equality in distribution), we get that

$$\begin{aligned} \frac{P'_n(z)}{P_n(z)} &= \frac{\xi_3 + i\xi_4}{\xi_1 + i\xi_2} \\ &\stackrel{\text{D}}{=} \frac{(l_{31} + il_{41})\zeta_1 + (l_{32} + il_{42})\zeta_2 + (l_{33} + il_{43})\zeta_3 + il_{44}\zeta_4}{(l_{11} + il_{21})\zeta_1 + il_{22}\zeta_2}. \end{aligned}$$

Hence, exploiting the independence of the  $\zeta_i$ 's, we see that

$$(12) \quad F(z) = \mathbf{E} \frac{P'_n(z)}{P_n(z)} = \mathbf{E} \frac{\alpha\zeta_1 + \beta\zeta_2}{\gamma\zeta_1 + \delta\zeta_2},$$

where

$$\begin{aligned} \alpha &= l_{31} + il_{41} & \beta &= l_{32} + il_{42} \\ \gamma &= l_{11} + il_{21} & \delta &= il_{22}. \end{aligned}$$

Splitting up the numerator in (31) and exploiting the exchangeability of  $\zeta_1$  and  $\zeta_2$ , we can rewrite the expectation as follows:

$$F(z) = \frac{\alpha}{\delta} f(\gamma/\delta) + \frac{\beta}{\gamma} f(\delta/\gamma),$$

where  $f$  is a complex-valued function defined on  $\mathbf{C} \setminus \mathbf{R}$  by

$$f(w) = \mathbf{E} \frac{\zeta_1}{w\zeta_1 + \zeta_2}.$$

The expectation appearing in the definition of  $f$  can be explicitly computed. Indeed,

$$\begin{aligned} f(w) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\rho \cos \phi}{w\rho \cos \phi + \rho \sin \phi} e^{-\rho^2/2} \rho d\rho d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{w + \tan \phi} \end{aligned}$$

and this last integral can be computed explicitly giving us

$$f(w) = \begin{cases} \frac{1}{w+i}, & \text{Im}(w) > 0, \\ \frac{1}{w-i}, & \text{Im}(w) < 0. \end{cases}$$

Recalling the definition of  $\delta$  and  $\gamma$ , we see that

$$\frac{\gamma}{\delta} = \frac{l_{21}}{l_{22}} - i \frac{l_{11}}{l_{22}}.$$

In general,  $l_{11}$  and  $l_{22}$  are just nonnegative. However, it is not hard to show that they are both strictly positive whenever  $z$  has a nonzero imaginary part. Hence,  $\gamma/\delta$  lies in the lower half-plane,  $\delta/\gamma$  lies in the upper half-plane, and

$$\begin{aligned} (13) \quad F(z) &= \frac{\alpha}{\delta} \frac{1}{\frac{\gamma}{\delta} - i} + \frac{\beta}{\gamma} \frac{1}{\frac{\delta}{\gamma} + i} \\ &= \frac{i\alpha + \beta}{i\gamma + \delta} \\ &= \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{-l_{21} + i(l_{11} + l_{22})}. \end{aligned}$$

At this point, we need explicit formulas for the elements of the Cholesky factor  $L$ . From (29) and (30), we see that

$$\begin{aligned} a^T a &= l_{11}^2 \\ b^T a &= l_{21}l_{11} & b^T b &= l_{21}^2 + l_{22}^2 \\ c^T a &= l_{31}l_{11} & c^T b &= l_{31}l_{21} + l_{32}l_{22} \\ d^T a &= l_{41}l_{11} & d^T b &= l_{41}l_{21} + l_{42}l_{22}. \end{aligned}$$

Solving these equations in succession, we get

$$\begin{aligned} l_{11} &= \frac{a^T a}{\sqrt{a^T a}} \\ l_{21} &= \frac{b^T a}{\sqrt{a^T a}} & l_{22} &= \frac{(a^T a)(b^T b) - (b^T a)^2}{\sqrt{a^T a}R} \\ l_{31} &= \frac{c^T a}{\sqrt{a^T a}} & l_{32} &= \frac{(a^T a)(c^T b) - (c^T a)(b^T a)}{\sqrt{a^T a}R} \\ l_{41} &= \frac{d^T a}{\sqrt{a^T a}} & l_{42} &= \frac{(a^T a)(d^T b) - (d^T a)(b^T a)}{\sqrt{a^T a}R} \end{aligned}$$

where

$$R = \sqrt{(a^T a)(b^T b) - (b^T a)^2}.$$

Substituting these expressions into (32) and simplifying, we see that

$$(14) \quad F(z) = \frac{-d^T a + ic^T a - i(a^T a(-d^T b + ic^T b) - (-d^T a + ic^T a)b^T a)/R}{-b^T a + ia^T a + iR}.$$

Recalling the definitions of  $a_j$ ,  $b_j$ ,  $c_j$ , and  $d_j$  given in (28), it is easy to check that the following identities hold:

$$\begin{aligned} a^T a &= \frac{1}{4}(A_0 + 2B_0 + \bar{A}_0), & b^T b &= -\frac{1}{4}(A_0 - 2B_0 + \bar{A}_0), \\ b^T a &= -\frac{i}{4}(A_0 - \bar{A}_0), & c^T b &= -\frac{i}{4}(A_1 - B_1 + \bar{B}_1 - \bar{A}_1), \\ c^T a &= \frac{1}{4}(A_1 + B_1 + \bar{B}_1 + \bar{A}_1), & d^T b &= -\frac{1}{4}(A_1 - B_1 - \bar{B}_1 + \bar{A}_1), \\ d^T a &= -\frac{i}{4}(A_1 + B_1 - \bar{B}_1 - \bar{A}_1), \end{aligned}$$

Plugging these expressions into (33) and simplifying, we get that

$$(15) \quad F(z) = \frac{A_1 + B_1 + (A_0 B_1 + B_0 B_1 - A_1 B_0 - \bar{A}_0 A_1)/D_0}{A_0 + B_0 + D_0},$$

where  $D_0$  is as given in (1). It turns out that further simplification occurs if we make the denominator real by the usual technique of multiplying and dividing by its complex conjugate. We leave out the algebraic details except to mention that a factor of  $A_0 + 2B_0 + \bar{A}_0$  cancels out from the numerator and denominator leaving us with

$$(16) \quad F(z) = \frac{B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1}{D_0 (B_0 + D_0)},$$

or, expanding out  $D_0^2$ ,

$$(17) \quad F(z) = \frac{B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1}{B_0 D_0 + B_0^2 - \bar{A}_0 A_0}.$$

□

LEMMA 4.3. *On the real axis,  $F$  has a jump discontinuity. Indeed, for each  $a \in \mathbb{R}$ ,*

$$\lim_{z \rightarrow a: \operatorname{Im}(z) > 0} F = \frac{B_1(a) - i E_1(a)}{B_0(a)}$$

and

$$\lim_{z \rightarrow a: \operatorname{Im}(z) < 0} F = \frac{B_1(a) + i E_1(a)}{B_0(a)}.$$

PROOF. Consider a point  $a$  on the real axis. On the reals,  $A_k = B_k$ , for  $k = 0, 1$ , and so  $D_0 = 0$ . Hence, the right-hand side in (16) is an indeterminate form. To analyze the limiting behavior of  $F$  near the real axis, we first divide the numerator and denominator by  $D_0$ :

$$(18) \quad F = \frac{B_1 + \frac{B_0 B_1 - \bar{A}_0 A_1}{D_0}}{B_0 + D_0}.$$

Now, only the ratio in the numerator is indeterminate. To study it, we start by expressing things in terms of the  $f_j$  functions:

$$\begin{aligned} B_0 B_1 - \bar{A}_0 A_1 &= \sum_{j,k} \overline{f_j(z)} f'_k(z) \left( f_j(z) \overline{f_k(z)} - \overline{f_j(z)} f_k(z) \right) \\ &= 2i \sum_{j,k} \overline{f_j(z)} f'_k(z) \operatorname{Im} \left( f_j(z) \overline{f_k(z)} \right) \end{aligned}$$

and

$$\begin{aligned} D_0^2 &= B_0^2 - |A_0|^2 = \sum_{j,k} f_j(z) \overline{f_k(z)} \left( \overline{f_j(z)} f_k(z) - f_j(z) \overline{f_k(z)} \right) \\ &= -2i \sum_{j,k} f_j(z) \overline{f_k(z)} \operatorname{Im} \left( f_j(z) \overline{f_k(z)} \right). \end{aligned}$$

Next, we write the first few terms of the Taylor series expansion of the  $f_j$ 's about the point  $z = a$ , substitute the expansions into the formulas above and then drop "high" order terms to derive the first few terms of the Taylor expansions for  $B_0B_1 - \bar{A}_0A_1$  and  $B_0^2 - |A_0|^2$ . For the first expression, we only need to go to linear terms to get

$$\begin{aligned} B_0B_1 - \bar{A}_0A_1 &= 2i \sum_{j,k} f_j(a)f'_k(a) \left( f'_j(a)f_k(a) - f_j(a)f'_k(a) \right) y + o(z-a) \\ &= 2i \left( A_1(a)^2 - A_0(a)A_2(a) \right) y + o(z-a) \end{aligned}$$

(as usual, we use  $y$  to denote the imaginary part of  $z$ ). For the second expression, we need to go to quadratic terms. The result is

$$\begin{aligned} D_0^2 &= 4 \sum_{j,k} \left( f'_j(a)^2 f_k(a)^2 - f_j(a)f'_j(a)f_k(a)f'_k(a) \right) y^2 + o((z-a)^2) \\ &= 4 \left( \left( \sum_{j=0}^n f'_j(a)^2 \right) \left( \sum_{j=0}^n f_j(a)^2 \right) - \left( \sum_{j=0}^n f'_j(a)f_j(a) \right)^2 \right) y^2 + o((z-a)^2) \\ &= 4 \left( A_2(a)A_0(a) - A_1(a)^2 \right) y^2 + o((z-a)^2). \end{aligned}$$

Hence, we see that

$$(19) \quad \frac{B_0B_1 - \bar{A}_0A_1}{D_0} = -i E_1(a) \operatorname{sgn}(a) \operatorname{sgn}(y) + o(z-a).$$

Combining (18) and (19), we get the desired limits expressing the jump discontinuity on the real axis.  $\square$

*Proof of Theorem 1.1.* Without loss of generality, it suffices to consider regions  $\Omega$  that are either regions that do not intersect the real axis or small rectangles centered on the real axis. We begin by considering a region  $\Omega$  that does not intersect the real axis. Applying Stokes' theorem to the expression for  $\mathbf{E}\nu_n(\Omega)$  given in Proposition 5.1, we see that

$$\mathbf{E}\nu_n(\Omega) = \frac{1}{\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} F(z, \bar{z}) dx dy.$$

Note that we are now writing  $F(z, \bar{z})$  to emphasize the fact that  $F$  depends on both  $z$  and  $\bar{z}$ . Letting the dagger symbol stand for the derivative with

respect to  $\bar{z}$ , we see from Lemma 5.2 that

$$(20) \quad \frac{\partial F}{\partial \bar{z}} = \left\{ (B_0 D_0 + B_0^2 - \bar{A}_0 A_0)(B_1^\dagger D_0 + B_1 D_0^\dagger + B_0^\dagger B_1 + B_0 B_1^\dagger - \bar{A}_0^\dagger A_1) \right. \\ \left. - (B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1)(B_0^\dagger D_0 + B_0 D_0^\dagger + 2B_0 B_0^\dagger - \bar{A}_0^\dagger A_0) \right\} \\ / (B_0 D_0 + B_0^2 - |A_0|^2)^2.$$

Recall that we have assumed that the functions  $f_j$  are entire and are real-valued on the real line. Hence, they have the property that  $\overline{f_j(z)} = f_j(\bar{z})$ . Their derivatives also have this property. Exploiting these facts, it is easy to check that

$$(21) \quad B_0^\dagger = \bar{B}_1, \quad \bar{A}_0^\dagger = 2\bar{A}_1, \quad B_1^\dagger = B_2.$$

Recalling that  $D_0 = \sqrt{B_0^2 - |A_0|^2}$ , we get that

$$(22) \quad D_0^\dagger = \frac{B_0 \bar{B}_1 - A_0 \bar{A}_1}{D_0}.$$

As explained in [22], substituting these formulas for the derivatives into the expression given above for  $\partial F / \partial \bar{z}$  followed by careful algebraic simplifications (see the appendix for the details) eventually leads to the fact that  $(1/\pi)\partial F(z, \bar{z})/\partial \bar{z}$  equals the expression given for  $h_n$  in the statement of the theorem.  $\square$

*Proof of Theorem 1.2.* Consider a narrow rectangle that straddles an interval of the real axis:  $\Omega = [a, b] \times [-\varepsilon, \varepsilon]$  where  $a < b$  and  $\varepsilon > 0$ . Writing the contour integral for  $\mathbf{E}\nu_n(\Omega)$  given by Proposition 5.1 and letting  $\varepsilon$  tend to 0, we see that

$$\mathbf{E}\nu_n((a, b)) = \frac{1}{2\pi i} \int_a^b (F(x-) - F(x+)) dx,$$

where  $\nu_n((a, b))$  denotes the number of zeros in the interval  $(a, b)$  of the real axis and

$$F(x-) = \lim_{z \rightarrow x: \text{Im}(z) < 0} F(z) \quad \text{and} \quad F(x+) = \lim_{z \rightarrow x: \text{Im}(z) > 0} F(z).$$

From Lemma 4.3, we see that

$$g_n(x) = \frac{1}{2\pi i} (F(x-) - F(x+)) = \frac{E_1(x)}{\pi B_0}.$$

This completes the proof.  $\square$

**5. The Intensity Functions  $h_n$  and  $g_n$ .** This section is devoted to the proof of Theorems 1.1 and 1.2. We begin with the following proposition.

PROPOSITION 5.1. *For each set  $\Omega \subset \mathbf{C}$  whose boundary intersects the set  $\{z \mid D_0(z) = 0\}$  at most only finitely many times,*

$$(23) \quad \mathbf{E}\nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} F(z) dz,$$

where

$$(24) \quad F = \frac{B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1}{B_0 D_0 + B_0^2 - \bar{A}_0 A_0}.$$

PROOF. The argument principle (see, e.g., [1], p. 151) gives an explicit formula for the random variable  $\nu_n(\Omega)$ , namely

$$(25) \quad \nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{P'_n(z)}{P_n(z)} dz.$$

Taking expectations in (25) and then interchanging expectation and contour integration (the justification of which is tedious but doable), we get

$$(26) \quad \mathbf{E}\nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \mathbf{E} \frac{P'_n(z)}{P_n(z)} dz.$$

The following Lemma shows that, away from the set  $\{z \mid D_0(z) = 0\}$ , the function

$$(27) \quad F(z) = \mathbf{E} \frac{P'_n(z)}{P_n(z)}$$

simplifies to the expression given in (24) and, since we've assumed that  $\partial\Omega$  intersects this set at only finitely many points, this finishes the proof.  $\square$

LEMMA 5.2. *Let  $F$  denote the function defined by (27). For  $z \notin \{z \mid D_0(z) = 0\}$ ,*

$$F(z) = \frac{A_1 + 3B_1 + \bar{B}_1 - \bar{A}_1}{4B_0}.$$

PROOF. Note that  $P_n(z)$  and  $P'_n(z)$  are complex Gaussian random variables. It is convenient to work with their real and imaginary parts,

$$\begin{aligned} P_n(z) &= \xi_1 + i\xi_2, \\ P'_n(z) &= \xi_3 + i\xi_4, \end{aligned}$$

which are just linear combinations of the original standard normal random variables:

$$\begin{aligned}\xi_1 &= \sum_{j=0}^n (a_j \eta_j - b_j \theta_j), & \xi_2 &= \sum_{j=0}^n (a_j \theta_j + b_j \eta_j), \\ \xi_3 &= \sum_{j=0}^n (c_j \eta_j - d_j \theta_j), & \xi_4 &= \sum_{j=0}^n (c_j \theta_j + d_j \eta_j).\end{aligned}$$

The coefficients in these linear combinations are given by

$$(28) \quad \begin{aligned}a_j &= \operatorname{Re}(f_j(z)) = \frac{f_j(z) + \overline{f_j(z)}}{2}, \\ b_j &= \operatorname{Im}(f_j(z)) = \frac{f_j(z) - \overline{f_j(z)}}{2i}, \\ c_j &= \operatorname{Re}(f'_j(z)) = \frac{f'_j(z) + \overline{f'_j(z)}}{2}, \\ d_j &= \operatorname{Im}(f'_j(z)) = \frac{f'_j(z) - \overline{f'_j(z)}}{2i}.\end{aligned}$$

Put  $\xi = [\xi_1 \ \xi_2 \ \xi_3 \ \xi_4]^T$ . The covariance among these four Gaussian random variables is easy to compute:

$$(29) \quad \operatorname{Cov}(\xi) = \mathbf{E}\xi\xi^T = \begin{bmatrix} a^T a + b^T b & 0 & a^T c + b^T d & a^T d - b^T c \\ 0 & a^T a + b^T b & b^T c - a^T d & a^T c + b^T d \\ a^T c + b^T d & b^T c - a^T d & c^T c + d^T d & 0 \\ a^T d - b^T c & a^T c + b^T d & 0 & c^T c + d^T d \end{bmatrix}$$

We now represent these four correlated Gaussian random variables in terms of four independent standard normals. To this end, we seek a lower triangular matrix  $L = [l_{ij}]$  such that the vector  $\xi$  is equal in distribution to  $L\zeta$ , where  $\zeta = [\zeta_1 \ \zeta_2 \ \zeta_3 \ \zeta_4]^T$  is a vector of four independent standard normal random variables. The following simple calculation shows that  $L$  is the Cholesky factor for the covariance matrix:

$$(30) \quad \operatorname{Cov}(\xi) = \mathbf{E}\xi\xi^T = \mathbf{E}L\zeta\zeta^T L^T = LL^T.$$

Now, since  $\xi \stackrel{\text{D}}{=} L\zeta$  and  $L$  is lower triangular (the symbol  $\stackrel{\text{D}}{=}$  denotes equality in distribution), we get that

$$\begin{aligned}\frac{P'_n(z)}{P_n(z)} &= \frac{\xi_3 + i\xi_4}{\xi_1 + i\xi_2} \\ &\stackrel{\text{D}}{=} \frac{(l_{31} + il_{41})\zeta_1 + (l_{32} + il_{42})\zeta_2 + (l_{33} + il_{43})\zeta_3 + il_{44}\zeta_4}{(l_{11} + il_{21})\zeta_1 + il_{22}\zeta_2}.\end{aligned}$$

Hence, exploiting the independence of the  $\zeta_i$ 's, we see that

$$(31) \quad F(z) = \mathbf{E} \frac{P'_n(z)}{P_n(z)} = \mathbf{E} \frac{\alpha \zeta_1 + \beta \zeta_2}{\gamma \zeta_1 + \delta \zeta_2},$$

where

$$\begin{aligned} \alpha &= l_{31} + il_{41} & \beta &= l_{32} + il_{42} \\ \gamma &= l_{11} + il_{21} & \delta &= il_{22}. \end{aligned}$$

Splitting up the numerator in (31) and exploiting the exchangeability of  $\zeta_1$  and  $\zeta_2$ , we can rewrite the expectation as follows:

$$F(z) = \frac{\alpha}{\delta} f(\gamma/\delta) + \frac{\beta}{\gamma} f(\delta/\gamma),$$

where  $f$  is a complex-valued function defined on  $\mathbf{C} \setminus \mathbf{R}$  by

$$f(w) = \mathbf{E} \frac{\zeta_1}{w\zeta_1 + \zeta_2}.$$

The expectation appearing in the definition of  $f$  can be explicitly computed. Indeed,

$$\begin{aligned} f(w) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\rho \cos \phi}{w\rho \cos \phi + \rho \sin \phi} e^{-\rho^2/2} \rho d\rho d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{w + \tan \phi} \end{aligned}$$

and this last integral can be computed explicitly giving us

$$f(w) = \begin{cases} \frac{1}{w+i}, & \text{Im}(w) > 0, \\ \frac{1}{w-i}, & \text{Im}(w) < 0. \end{cases}$$

Recalling the definition of  $\delta$  and  $\gamma$ , we see that

$$\frac{\gamma}{\delta} = \frac{l_{21}}{l_{22}} - i \frac{l_{11}}{l_{22}}.$$

In general,  $l_{11}$  and  $l_{22}$  are just nonnegative. However, it is not hard to show that they are both strictly positive whenever  $z$  has a nonzero imaginary part. Hence,  $\gamma/\delta$  lies in the lower half-plane,  $\delta/\gamma$  lies in the upper half-plane, and

$$\begin{aligned} (32) \quad F(z) &= \frac{\alpha}{\delta} \frac{1}{\frac{\gamma}{\delta} - i} + \frac{\beta}{\gamma} \frac{1}{\frac{\delta}{\gamma} + i} \\ &= \frac{i\alpha + \beta}{i\gamma + \delta} \\ &= \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{-l_{21} + i(l_{11} + l_{22})}. \end{aligned}$$

At this point, we need explicit formulas for the elements of the Cholesky factor  $L$ . From (29) and (30), we see that

$$\begin{aligned} a^T a + b^T b &= l_{11}^2 \\ 0 &= l_{21} l_{11} & a^T a + b^T b &= l_{21}^2 + l_{22}^2 \\ a^T c + b^T d &= l_{31} l_{11} & b^T c - a^T d &= l_{31} l_{21} + l_{32} l_{22} \\ a^T d - b^T c &= l_{41} l_{11} & a^T c + b^T d &= l_{41} l_{21} + l_{42} l_{22}. \end{aligned}$$

Solving these equations in succession, we get

$$\begin{aligned} l_{11} &= \frac{a^T a + b^T b}{\sqrt{a^T a + b^T b}} \\ l_{21} &= 0 & l_{22} &= \frac{a^T a + b^T b}{\sqrt{a^T a + b^T b}} \\ l_{31} &= \frac{a^T c + b^T d}{\sqrt{a^T a + b^T b}} & l_{32} &= \frac{b^T c - a^T d}{\sqrt{a^T a + b^T b}} \\ l_{41} &= \frac{a^T d - b^T c}{\sqrt{a^T a + b^T b}} & l_{42} &= \frac{a^T c + b^T d}{\sqrt{a^T a + b^T b}}. \end{aligned}$$

Substituting these expressions into (32) and simplifying, we see that

$$(33) \quad F(z) = \frac{b^T c - a^T d + i(a^T c + b^T d)}{i(a^T a + b^T b)}.$$

Recalling the definitions of  $a_j$ ,  $b_j$ ,  $c_j$ , and  $d_j$  given in (28), it is easy to check that the following identities hold:

$$\begin{aligned} a^T a &= \frac{1}{4}(A_0 + 2B_0 + \bar{A}_0), & b^T b &= -\frac{1}{4}(A_0 - 2B_0 + \bar{A}_0), \\ a^T b &= -\frac{i}{4}(A_0 - \bar{A}_0), & b^T c &= -\frac{i}{4}(A_1 - B_1 + \bar{B}_1 - \bar{A}_1), \\ a^T c &= \frac{1}{4}(A_1 + B_1 + \bar{B}_1 + \bar{A}_1), & b^T d &= -\frac{1}{4}(A_1 - B_1 - \bar{B}_1 + \bar{A}_1). \\ a^T d &= -\frac{i}{4}(A_1 + B_1 - \bar{B}_1 - \bar{A}_1), \end{aligned}$$

Plugging these expressions into (33) and simplifying, we get that

$$(34) \quad F(z) = \frac{B_1}{B_0}.$$

□

*Proof of Theorem 1.1.* Without loss of generality, it suffices to consider regions  $\Omega$  that are either regions that do not intersect the real axis or small rectangles centered on the real axis. We begin by considering a region  $\Omega$  that does not intersect the real axis. Applying Stokes' theorem to the expression for  $\mathbf{E}\nu_n(\Omega)$  given in Proposition 5.1, we see that

$$\mathbf{E}\nu_n(\Omega) = \frac{1}{\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} F(z, \bar{z}) dx dy.$$

Note that we are now writing  $F(z, \bar{z})$  to emphasize the fact that  $F$  depends on both  $z$  and  $\bar{z}$ . Letting the dagger symbol stand for the derivative with respect to  $\bar{z}$ , we see from Lemma 5.2 that

$$(35) \quad \frac{\partial F}{\partial \bar{z}} = \frac{B_0 B_1^\dagger - B_1 B_0^\dagger}{B_0^2}.$$

Recall that we have assumed that the functions  $f_j$  are entire and are real-valued on the real line. Hence, they have the property that  $\overline{f_j(z)} = f_j(\bar{z})$ . Their derivatives also have this property. Exploiting these facts, it is easy to check that

$$(36) \quad B_0^\dagger = \bar{B}_1, \quad B_1^\dagger = B_2,$$

Plugging in, we get

$$(37) \quad \frac{\partial F}{\partial \bar{z}} = \frac{B_0 B_2 - |B_1|^2}{B_0^2}.$$

Hence,  $(1/\pi)\partial F(z, \bar{z})/\partial \bar{z}$  equals the expression given for  $h_n$  in the statement of the theorem.  $\square$

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APPENDIX A: ALGEBRAIC SIMPLIFICATION OF THE FORMULA  
FOR  $\partial F/\partial \bar{Z}$

Substituting the derivatives given in (37) and (22) into the formula (35) for  $\partial F/\partial \bar{z}$ , we get that the denominator simplifies to

$$\begin{aligned} \text{denom} \left( \frac{\partial F}{\partial \bar{z}} \right) &= (B_0 D_0 + B_0^2 - \bar{A}_0 A_0)^2 \\ &= (B_0 D_0 + D_0^2)^2 \\ &= (B_0 + D_0)^2 D_0^2. \end{aligned}$$

and the numerator of the formula becomes

$$\begin{aligned} \text{num} \left( \frac{\partial F}{\partial \bar{z}} \right) &= (B_0 D_0 + B_0^2 - \bar{A}_0 A_0) \left( B_2 D_0 + B_1 \frac{B_0 \bar{B}_1 - A_0 \bar{A}_1}{D_0} \right. \\ &\quad \left. + \bar{B}_1 B_1 + B_0 B_2 - 2 \bar{A}_1 A_1 \right) \\ &\quad - (B_1 D_0 + B_0 B_1 - \bar{A}_0 A_1) \left( \bar{B}_1 D_0 + B_0 \frac{B_0 \bar{B}_1 - A_0 \bar{A}_1}{D_0} \right. \\ &\quad \left. + 2 B_0 \bar{B}_1 - 2 \bar{A}_1 A_0 \right). \end{aligned}$$

The first step to simplifying the numerator is to replace  $B_0^2 - \bar{A}_0 A_0$  in the first term with  $D_0$  (like we did in the denominator) and factor out a  $1/D_0$  to get

$$\begin{aligned} \text{num} \left( \frac{\partial F}{\partial \bar{z}} \right) &= \frac{1}{D_0} (B_0 + D_0) D_0 \left( B_2 D_0^2 + B_1 B_0 \bar{B}_1 - B_1 A_0 \bar{A}_1 \right. \\ &\quad \left. + \bar{B}_1 B_1 D_0 + B_0 B_2 D_0 - 2 \bar{A}_1 A_1 D_0 \right) \\ &\quad - \frac{1}{D_0} ((B_0 + D_0) B_1 - \bar{A}_0 A_1) \left( \bar{B}_1 D_0^2 + B_0 B_0 \bar{B}_1 - B_1 A_0 \bar{A}_1 \right. \\ &\quad \left. + 2 B_0 \bar{B}_1 D_0 - 2 \bar{A}_1 A_0 D_0 \right). \end{aligned}$$

Next, we bundle together the terms that have a  $B_0 + D_0$  factor:

$$\begin{aligned} \text{num} \left( \frac{\partial F}{\partial \bar{z}} \right) &= \frac{1}{D_0} (B_0 + D_0) \left( B_2 D_0^3 + B_1 B_0 \bar{B}_1 D_0 - B_1 A_0 \bar{A}_1 D_0 \right. \\ &\quad \left. + \bar{B}_1 B_1 D_0^2 + B_0 B_2 D_0^2 - 2 \bar{A}_1 A_1 D_0^2 \right. \\ &\quad \left. - |B_1|^2 D_0^2 - B_0 B_0 |B_1|^2 + B_1^2 A_0 \bar{A}_1 \right. \\ &\quad \left. - 2 B_0 |B_1|^2 D_0 + 2 \bar{A}_1 A_0 B_1 D_0 \right) \\ &\quad + \frac{1}{D_0} \bar{A}_0 A_1 \left( \bar{B}_1 D_0^2 + B_0 B_0 \bar{B}_1 - B_1 A_0 \bar{A}_1 \right. \\ &\quad \left. + 2 B_0 \bar{B}_1 D_0 - 2 \bar{A}_1 A_0 D_0 \right). \end{aligned}$$

Now, there are several places where we can find  $B_0 + D_0$  factors. For example, the big factor containing eleven terms can be rewritten as follows:

$$\begin{aligned} &B_2 D_0^3 + B_1 B_0 \bar{B}_1 D_0 - B_1 A_0 \bar{A}_1 D_0 + \bar{B}_1 B_1 D_0^2 + B_0 B_2 D_0^2 - 2 \bar{A}_1 A_1 D_0^2 \\ &- |B_1|^2 D_0^2 - B_0 B_0 |B_1|^2 + B_1^2 A_0 \bar{A}_1 - 2 B_0 |B_1|^2 D_0 + 2 \bar{A}_1 A_0 B_1 D_0 \\ &= (B_0 + D_0) (B_2 D_0^2 - B_0 |B_1|^2 + A_0 \bar{A}_1 B_1) - 2 |A_1|^2 D_0^2. \end{aligned}$$

We also look for  $B_0 + D_0$  factors in the five-term factor:

$$\begin{aligned} &\bar{B}_1 D_0^2 + B_0 B_0 \bar{B}_1 - B_1 A_0 \bar{A}_1 + 2 B_0 \bar{B}_1 D_0 - 2 \bar{A}_1 A_0 D_0 \\ &= (B_0 + D_0) ((B_0 + D_0) \bar{B}_1 - A_0 \bar{A}_1) - A_0 \bar{A}_1 D_0. \end{aligned}$$

Substituting these expressions into the formula for the numerator, we get

$$\begin{aligned} D_0 \text{num} \left( \frac{\partial F}{\partial \bar{z}} \right) &= (B_0 + D_0) \left( (B_0 + D_0) (B_2 D_0^2 - B_0 |B_1|^2 + A_0 \bar{A}_1 B_1) - 2 |A_1|^2 D_0^2 \right) \\ &\quad + \bar{A}_0 A_1 \left( (B_0 + D_0) ((B_0 + D_0) \bar{B}_1 - A_0 \bar{A}_1) - A_0 \bar{A}_1 D_0 \right). \end{aligned}$$

Rearranging the terms, we see that

$$\begin{aligned} D_0 \text{num} \left( \frac{\partial F}{\partial \bar{z}} \right) &= (B_0 + D_0)^2 (B_2 D_0^2 - B_0 |B_1|^2 + A_0 \bar{A}_1 B_1 + \bar{A}_0 A_1 \bar{B}_1) \\ &\quad - (B_0 + D_0) \left( 2 |A_1|^2 D_0^2 + |A_0|^2 |A_1|^2 \right) \\ &\quad - |A_0|^2 |A_1|^2 D_0. \end{aligned}$$

Here's the tricky part... replace  $D_0^2$  in the second row with  $B_0^2 - |A_0|^2$  and the second and third row simplify nicely:

$$\begin{aligned}
& (B_0 + D_0) \left( 2|A_1|^2 D_0^2 + |A_0|^2 |A_1|^2 \right) + |A_0|^2 |A_1|^2 D_0 \\
&= (B_0 + D_0) \left( 2|A_1|^2 B_0^2 - |A_0|^2 |A_1|^2 \right) + |A_0|^2 |A_1|^2 D_0 \\
&= 2|A_1|^2 B_0^3 - |A_0|^2 |A_1|^2 B_0 + 2|A_1|^2 B_0^2 D_0 \\
&= |A_1|^2 B_0 \left( 2B_0^2 - |A_0|^2 + 2B_0 D_0 \right) \\
&= |A_1|^2 B_0 (B_0 + D_0)^2.
\end{aligned}$$

Substituting this expression into our formula for the numerator, we now have

$$D_0 \operatorname{num} \left( \frac{\partial F}{\partial \bar{z}} \right) = (B_0 + D_0)^2 (B_2 D_0^2 - B_0 (|A_1|^2 + |B_1|^2) + A_0 \bar{A}_1 B_1 + \bar{A}_0 A_1 \bar{B}_1).$$

Finally, we get a simple formula for  $\partial F / \partial \bar{z}$ :

$$\frac{\partial F}{\partial \bar{z}} = \frac{B_2 D_0^2 - B_0 (|A_1|^2 + |B_1|^2) + A_0 \bar{A}_1 B_1 + \bar{A}_0 A_1 \bar{B}_1}{D_0^3}.$$

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