Lagrange Points for Eccentric Planar 3-Body Systems

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ABSTRACT

Subject headings: Celestial mechanics

1. Introduction

Consider the planar 3-body problem. Let $z_j(t)$ denote the position vector for body j at time t and let m_j denote the mass of body j. Without loss of generality, we may assume that the center of mass of the system is the origin of the coordinate system:

$$\sum_{j=0}^{2} m_j z_j = 0. \tag{1}$$

We are interested in solutions in which the three bodies form an equilateral triangle at all times. That is,

$$R(t) := |z_1(t) - z_0(t)| = |z_2(t) - z_1(t)| = |z_0(t) - z_2(t)|.$$

In what follows, we usually omit showing the dependence of various quantities on time t.

The gravitational force on body j is given by

$$\begin{aligned} F_j &= Gm_j \sum_{k\neq j} m_k \frac{z_k - z_j}{|z_k - z_j|^3} \\ &= \frac{G}{R^3} m_j \sum_{k\neq j} m_k (z_k - z_j) \\ &= -\frac{G}{R^3} m_j M z_j, \end{aligned}$$

where $M = \sum_k m_k$ and the last equality follows from (1). Hence, the acceleration of body j is given as

$$\begin{aligned} \ddot{z}_j &= -\frac{G}{R^3} M z_j \\ &= -\frac{G}{R^3} M |z_j|^3 \frac{z_j}{|z_j|^3}. \end{aligned}$$

We seek solutions in which $|z_j(t)|/R(t)$ does not depend on t. Hence,

$$\ddot{z}_j = -\gamma_j \frac{z_j}{|z_j|^3} \tag{2}$$

where

$$\gamma_j = GM \left(\frac{|z_j(0)|}{R(0)} \right)^3. \tag{3}$$

The solutions to (2) are well known. If we write z_j in complex polar notation, $z_j = r_j e^{i\theta_j}$, then for each $-1 \leq e \leq 1$ there is an elliptical solution given implicitly by

$$r_j(\theta_j) = \frac{p_j}{1 + e\cos(\theta_j - \theta_j(0))} \tag{4}$$

and

$$\dot{\theta}_j = \frac{\sqrt{p_j \gamma_j}}{r_j^2}. \tag{5}$$

The velocity is easily obtained by differentiation:

$$\dot{z}_j = \left(\frac{dr_j}{d\theta_j} + i r_j \right) \dot{\theta}_j e^{i\theta_j}. \tag{6}$$

From (4), we have that

$$\frac{dr_j}{d\theta_j} = \frac{p_j e \sin(\theta_j - \theta_j(0))}{(1 + e\cos(\theta_j - \theta_j(0))^2}. \tag{7}$$

At $t = 0$, $\theta_j = \theta_j(0)$ and each mass is at periapsis (at least for $e > 0$). Hence, $dr_j/d\theta_j = 0$ and so

$$\dot{z}_j|_{\theta_j=\theta_j(0)} = ir_j\dot{\theta}_j e^{i\theta_j(0)}. \tag{8}$$

In words, the speed of the particle is $r_j\dot{\theta}_j$ and the direction is perpendicular to the line from the center of mass to mass j.

The simulator is therefore initialized as follows. We fix an initial separation value $R(0)$ and place the three masses at the vertices of an equilateral triangle of side $R(0)$. The coordinate system is shifted so that the origin is at the center of mass. In this coordinate system, we have $z_j(0) = r_{j,0}e^{i\theta_j(0)}$. The constants γ_j are computed according to (3). The constants p_j are computed using (4):

$$p_j = r_{j,0}(1 + e). \tag{9}$$

From these parameter calculations, one finally computes initial values for the velocities according to (8).