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LAGRANGE POINTS L_1, L_2 AND L_3

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1. TWO BODIES

Let R denote the Sun-Earth distance. Let r denote the Earth-L_2 distance. Let M denote the mass of the Sun. Let m denote the mass of the Earth. Let ρ denote the distance from the center of the Sun to the center of mass of the system. Let

$$z_E(t) = (R - \rho)e^{2\pi it/T}$$

denote Earth's orbit about the center of mass of the system and let

$$z_S(t) = -\rho e^{2\pi it/T}$$

denote the orbit of the Sun about the center of mass. The fact that the center of mass is at the origin in our coordinate system leads to

$$mz_E(t) + Mz_S(t) = 0$$

which simplifies to

$$m(R - \rho) = M\rho$$

which we can solve for ρ:

$$\rho = \frac{m}{m + M}R$$

Earth's orbit satisfies Newton's law of motion:

$$mz_E''(t) = -GmM(z_E - z_S)/|z_E - z_S|^3$$

Of course, we can solve for T in terms of the masses and R using

$$z_E''(t) = -\frac{4\pi^2}{T^2}z_E(t) = -\frac{4\pi^2}{T^2}\frac{R-\rho}{R}(z_E(t) - z_S(t)) = -\frac{4\pi^2}{T^2}\frac{M}{M+m}(z_E(t) - z_S(t))$$

We get

$$\frac{4\pi^2}{T^2}\frac{M}{M+m} = GM/R^3.$$

Hence,

$$T = 2\pi\sqrt{\frac{R^3}{G(M+m)}}.$$

2. THIRD BODY

Now we consider a third body with infinitesimally small mass. We will call it the Lagrange body. Let

$$z_L(t) = (R - \rho + r)e^{2\pi it/T}$$

denote its orbit. If we assume that r is a small positive constnat, then this new body can be thought of as orbiting at the Sun/Earth L_2 point. But, if r is negative, this body could be viewed as orbiting at L_1 or even at L_3. And, if we let r be complex valued, then this body could be at L4 or L5. The L_2 orbit must also satisfy Newton's law of motion:

$$z_L''(t) = -GM(z_L - z_S)/|z_L - z_S|^3 - Gm(z_L - z_E)/|z_L - z_E|^3.$$

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Again, we can simplify using

$$z_L''(t) = -\frac{4\pi^2}{T^2} z_L(t)$$

$$z_L - z_S = (R+r)e^{2\pi i t/T} = \frac{R+r}{R-\rho+r} z_L$$

and

$$z_L - z_E = r e^{2\pi i t/T} = \frac{r}{R-\rho+r} z_L.$$

We get

$$(M+m)\frac{1}{R^3} = M\frac{R+r}{R-\rho+r}\ \frac{1}{|R+r|^3} + m\frac{r}{R-\rho+r}\ \frac{1}{|r|^3}.$$

At this point things get tricky if r is not real because the length of a complex number involves the number, its conjugate and something to the $3/2$'s power. So, henceforth, let's assume that r is real. In this case, our formula simplifies to

$$(M+m)\frac{1}{R^3} = \varepsilon_1 M\frac{1}{R-\rho+r}\ \frac{1}{(R+r)^2} + \varepsilon_2 m\frac{1}{R-\rho+r}\ \frac{1}{r^2}$$

where $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$ when r is positive (the L_2 scenario), $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$ when r is negative but $R+r$ is positive (the L_1 scenario) and $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$ when both r and $R+r$ are negative (the L_3 scenario). Cross multiplying, we get

$$(M+m)r^2(R-\rho+r)(R+r)^2 = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R+r)^2$$

Now, we use the fact that $R-\rho = \frac{M}{M+m}R$ to rewrite the equation as:

$$r^2(MR+(M+m)r)(R+r)^2 = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R+r)^2$$

Expanding the powers, we get

$$r^2MR(R^2+2Rr+r^2) + r^2(M+m)r(R^2+2Rr+r^2) = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R^2+2Rr+r^2)$$

Simplifying, we get

$$r^2MR\left((1-\varepsilon_1)R^2+2Rr+r^2\right) + r^2(M+m)r(R^2+2Rr+r^2) = \varepsilon_2 mR^3(R^2+2Rr+r^2)$$

Dividing by M and R^5 and letting $\mu = m/M$ and $x = r/R$, we get

$$x^2\left((1-\varepsilon_1)+2x+x^2\right) + x^2(1+\mu)x(1+2x+x^2) = \varepsilon_2\mu(1+2x+x^2)$$

Writing it as a simple polynomial in x, we get

$$(1+\mu)x^5 + (3+2\mu)x^4 + (3+\mu)x^3 + (1-\varepsilon_1-\varepsilon_2\mu)\,x^2 - 2\varepsilon_2\mu x - \varepsilon_2\mu = 0$$

In the Earth/Sun system, $\mu = 3.0\times 10^{-6}$. For each of the three scenarios (L_1, L_2, L_3), we used Python to find the roots to this 5-th degree polynomial. In each case, we find that four of the roots are complex. The sole real root is root of interest. Here's what we get:

Scenario	$1+x$
L_1	0.99003345
L_2	1.01003322
L_3	-0.99999825

For the L_2 case, our answer is in pretty close agreement to the value one finds on Wikipedia: $1+x = 1.01004$.