OPTIMAL SWITCHING AMONG SEVERAL BROWNIAN MOTIONS

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ABSTRACT. For $i = 1, \dots, d$, let $B^i_{s_i}$ be a one-dimensional Brownian motion on the interval $[0, a_i]$ with absorption at the endpoints. At each instant in time, we must decide to run some subset of these d Brownian motions while holding the others fixed at their current state. The resulting process evolves in the rectangle $D = [0, a_1] \times \cdots \times [0, a_d]$. If, at some instant, we decide to freeze all of the Brownian motions, then a reward is received in accordance with this final position. We consider two types of reward functions.

First, we assume that the reward is zero everywhere in D except along the d edges that correspond to the coordinate axes. Along these edges it is given by C^3 strictly-concave functions $\gamma_i(x_i)$ which are zero at the endpoints 0 and a_i of their domains. The optimal control for this problem has a simple description. Let

$$\Gamma_i(x_i) = -\int_0^{x_i} u\gamma_i''(u)du$$

and put

$$M_i = \{x \in D : \Gamma_i(x_i) = \max_j \Gamma_j(x_j)\}.$$

We prove that the optimal control is: on M_i run any Brownian motion except the i^{th} one and stop at the first time an edge is reached.

The second class of reward functions are assumed to be zero everywhere except on the facets of D that meet at the origin. On the i^{th} such facet (i.e., where $x_i = 0$), the reward function is the product of $\gamma_j(x_j)$ for $j \neq i$. Put

$$N_i = \{x \in D : \Gamma_i(x_i) = \min_j \Gamma_j(x_j)\}.$$

The optimal control is: on N_i run the i^{th} Brownian motion and stop when a facet of D is reached.

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Abbreviated title. Optimal Switching Among Brownian Motions

1. INTRODUCTION.

For $i = 1, \dots, d$, let $B^i = \{B^i_{s_i}, s_i \geq 0\}$ be a one-dimensional Brownian motion on the interval $[0, a_i]$ with absorption at the endpoints. We assume that B^i is adapted to a filtration $\mathcal{F}^i = \{\mathcal{F}^i_{s_i}, s_i \geq 0\}$ on the space Ω^i of continuous functions. Let $P^i_{x_i}$, $x_i \in [0, a_i]$, denote the probability measure associated with B^i starting at the point x_i and let $\mathbf{E}^i_{x_i}$ denote the corresponding expectation operator. We assume that the filtration $\mathcal{F}^i$ is complete with respect to every measure $P^i_{x_i}$, $x_i \in [0, a_i]$.

The problem we study involves switching between these Brownian motions. We take as our sample space, the product $\Omega = \Omega^1 \times \cdots \times \Omega^d$ and let the Brownian motions be independent by putting $P_x = P^1_{x_1} \times \cdots \times P^d_{x_d}$ ($\mathbf{E}_x$ will denote the corresponding expectation operator). A *switching strategy* T is a family of random d-tuples,

$$T = \{T(t) = (T_1(t), \ldots, T_d(t)),\ t \geq 0\}, \tag{1}$$

satisfying

$$T(0) = (0, \ldots, 0), \tag{2}$$

$$T_i(t) \text{ is increasing in } t \text{ for each } i \tag{3}$$

$$T_1(t) + \cdots + T_d(t) = t, \tag{4}$$

and

$$\{T_1(t) \leq s_1, \ldots, T_d(t) \leq s_d\} \in \mathcal{F}^1_{s_1} \times \cdots \times \mathcal{F}^d_{s_d}. \tag{5}$$

The random variable $T_i(t)$ represents the amount of time the i^{th} Brownian motion has been used up to time t. The interpretation of (4) is that, at time t, the total allocation of time between the d processes must equal t. Condition (5) says that the switching strategy must be non-anticipating. The *switched process* X^T is defined as

$$X^T(t) = B_{T(t)} = (B^1_{T_1(t)}, \ldots, B^d_{T_d(t)}). \tag{6}$$

There are several possible criteria which may be optimized. Perhaps the most common is the accumulated discounted reward. In this case, we assume that each Brownian motion has a running reward function $r_i(x_i)$ and the problem then is to find the strategy T^* which attains the following supremum:

$$v(x) = \sup_T \mathbf{E}_x \int_0^\infty e^{-\lambda t} r(X^T(t)) \cdot dT(t), \tag{7}$$

where λ is a fixed positive constant, $r(x) = (r_1(x_1), \ldots, r_d(x_d))$ and $r(X^T(t)) \cdot dT(t)$ represents the inner product between the vectors $r(X^T(t))$ and $dT(t)$. This problem was studied by Karatzas [2], Mandelbaum [3] and Dalang [1] as a continuous time generalization of Gittins' index theorem for Markov chains (see, e.g., [6] Chapter

14). Assuming each of the $r_i(x_i)$ are strictly increasing functions, they showed that there exist functions $\Gamma_i(x_i)$ which determine the optimal strategy as follows:

$$T_i^*(t) \text{ increases only when } X^{T^*}(t) \in M_i \tag{8}$$

where

$$M_i = \{x \in D : \Gamma_i(x_i) = \max_j \Gamma_j(x_j)\}. \tag{9}$$

This strategy is called a *follow the leader strategy* since it runs process i when $\Gamma_i(x_i)$ is the largest of all the functions $\Gamma_j(x_j)$. The functions $\Gamma_j(x_j)$ are called *index functions*.

A different optimization criterion was considered in [4]. For $d = 2$, we studied the problem of finding $T^*(t)$ which attains the following supremum:

$$v(x) = \sup_T \mathbf{E}_x f(X^T(\tau)), \tag{10}$$

where τ is the first time the switched process X^T exits a rectangle $D = [0, a_1] \times [0, a_2]$ and f is a continuous pay-off function defined on the edges of D and *strongly concave* (i.e. twice continuously differentiable and strictly concave) or linear on each edge. In the case where f is zero except on the two edges that meet at the origin, it turns out that the optimal strategy has a simple description. Indeed, let $\gamma_i(x_i)$ denote the restriction of f to the x_i coordinate axis and put

$$\Gamma_i(x_i) = -\int_0^{x_i} u\gamma_i''(u)du = \gamma_i(x_i) - x_i\gamma_i'(x_i). \tag{11}$$

In terms of the sets M_i defined in (9), the optimal strategy satisfies:

$$T_i^*(t) \text{ increases only when } X^{T^*}(t) \notin M_i.$$

Hence, the optimal strategy can be described as one which *follows the loser*.

The aim of this paper is to investigate how the above result generalizes to the case where $d > 2$. Two possibilities come to mind. First, we could put concave data on the one-dimensional faces (i.e. edges) of D that meet at the origin. In this case, we let τ be the first hitting time of the set of edges of D.

Theorem 1. *Let*

$$f(x) = \begin{cases} \gamma_1(x_1) & \text{if } x = (x_1, 0, \ldots, 0) \\ \vdots & \\ \gamma_d(x_d) & \text{if } x = (0, \ldots, 0, x_d) \\ 0 & \text{otherwise} \end{cases}$$

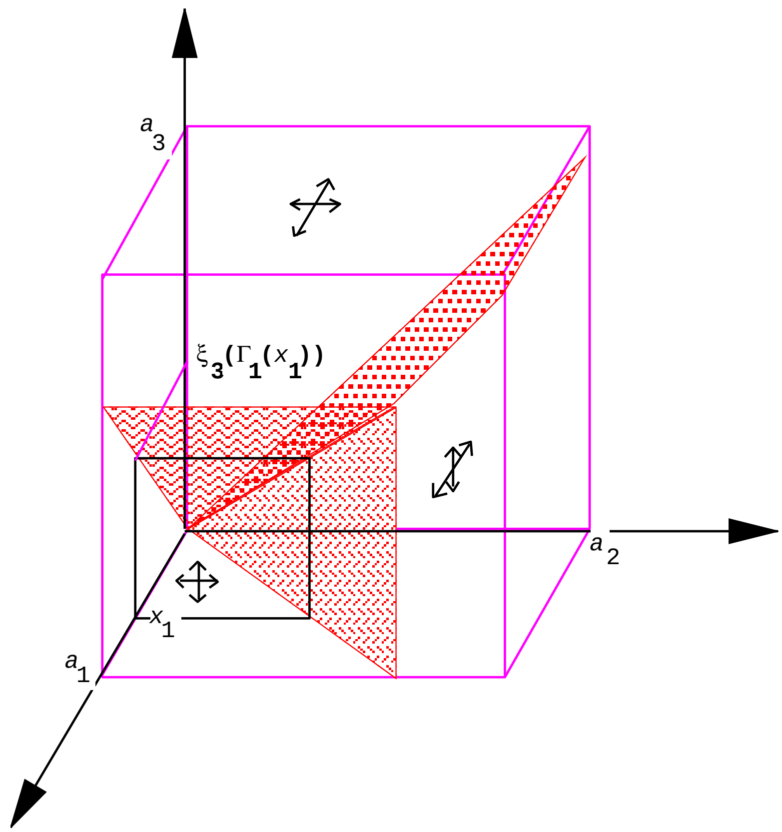

FIGURE 1. Following anybody but the leader. Here we show the switching surfaces in the special case where each are planar. The arrows indicate which Brownian motions can be run in each region.

where each $\gamma_i(x_i)$ is C^3, strictly-concave and vanishes at 0 and a_i. Optimal strategies exist. A strategy T^* is optimal if and only if

$$T_i^*(t) \text{ increases only when } X^{T^*}(t) \notin M_i \tag{12}$$

a.s. P_x, for all $x \in D$.

Hence, optimal strategies are ones which follow anybody but the leader. For $d = 3$, the control regions are shown in Figure 1 (in the case where the borders between the switching regions are planar).

Alternatively, we could put (certain types of) concave data on the codimension one faces (i.e. facets) of D that meet at the origin. In this case, τ is taken to be the first hitting time of a facet.

Theorem 2. *Let*

$$f(x) = \begin{cases} \gamma_2(x_2)\cdots\gamma_d(x_d) & \text{if } x = (0, x_2, \ldots, x_d) \\ \vdots & \\ \gamma_1(x_1)\cdots\gamma_{i-1}(x_{i-1})\gamma_{i+1}(x_{i+1})\cdots\gamma_d(x_d) & \text{if } x = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) \\ \vdots & \\ \gamma_1(x_1)\cdots\gamma_{d-1}(x_{d-1}) & \text{if } x = (x_1, \ldots, x_{d-1}, 0) \\ 0 & \text{otherwise} \end{cases}$$

where, for each i, $\gamma_i(x_i)$ is C^3, strictly-concave and vanishes at 0 and a_i. There exists a unique (up to almost sure equivalence) optimal strategy. It satisfies:

$$T_i^*(t) \text{ increases only when } X^{T^*}(t) \in N_i, \tag{13}$$

where

$$N_i = \{x \in D : \Gamma_i(x_i) = \min_j \Gamma_j(x_j)\}. \tag{14}$$

So, for facet-data, the optimal strategy follows the loser. For $d = 3$, the control regions are shown in Figure 2.

Remark: In Theorems 1 and 2 we assumed that the boundary data is three times continuously differentiable. Two derivatives should suffice. However, we will employ a change of variables in Sections 3 and 4 which necessitates our assumption of the existence of three derivatives. We believe that it should be possible to prove the results without using this change of variables, but the computations are more involved.

2. Probabilistic Preliminaries.

For the proofs of Theorems 1 and 2, we use the general theory of multiparameter processes. In this section, we review basic definitions and standard results. Our typical multiparameter process will be a real-valued function of $(B^1_{s_1}, \ldots, B^d_{s_d})$ and so will always be adapted to the multiparameter filtration $\mathcal{F} = \{\mathcal{F}^1_{s_1} \times \cdots \times \mathcal{F}^d_{s_d} : s_1 \geq 0, \ldots, s_d \geq 0\}$.

A multiparameter process $\mathcal{M}_{s_1,\ldots,s_d}$ is a *supermartingale* if it is adapted to $\mathcal{F}$, is integrable and satisfies the supermartingale property: for every $x \in D$ and for all $s_1 \leq t_1, \ldots, s_d \leq t_d$,

$$\mathbf{E}_x\{\mathcal{M}_{t_1,\ldots,t_d} | \mathcal{F}^1_{s_1} \times \cdots \times \mathcal{F}^d_{s_d}\} \leq \mathcal{M}_{s_1,\ldots,s_d}.$$

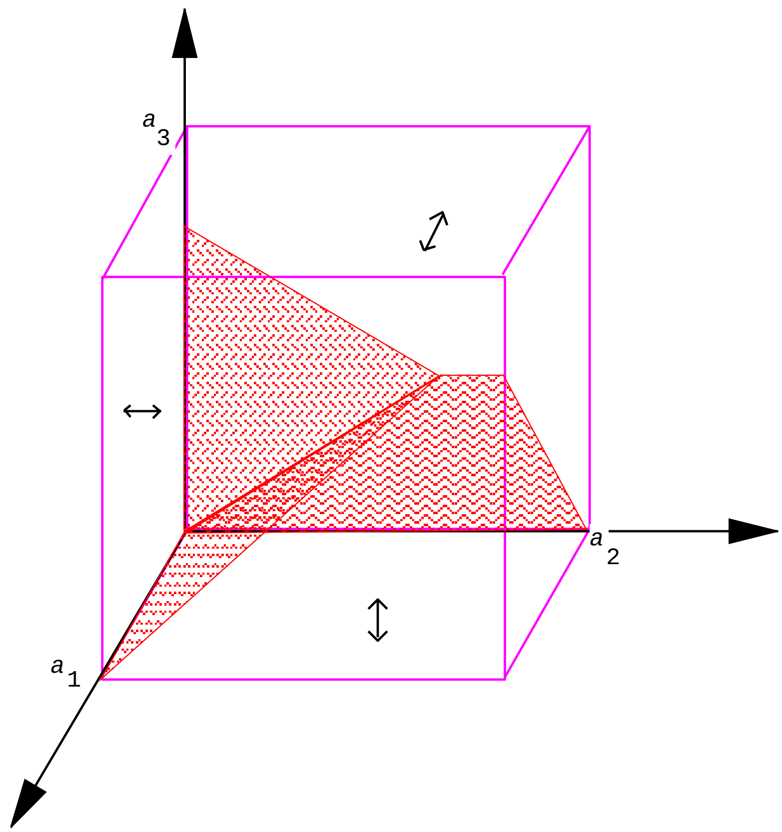

FIGURE 2. Following the loser. Here we show the switching surfaces in the special case where each are planar. The arrows indicate which Brownian motions can be run in each region.

It is a *martingale* if the above inequality is replaced by equality.

Associated with any switching strategy $T(t)$, there is a one-parameter filtration $\mathcal{F}^T = \{\mathcal{F}_{T(t)} : \ t \geq 0\}$ where $\mathcal{F}_{T(t)}$ is defined as the σ-algebra containing all measurable sets C for which $C \cap \{T_1(t) \leq s_1, \ldots, T_d(t) \leq s_d\} \in \mathcal{F}^1_{s_1} \times \cdots \times \mathcal{F}^d_{s_d}$ for all $s_1, \ldots, s_d$. The switched process $X^T(t)$ is adapted to $\mathcal{F}^T$.

When we say that a multiparameter process is a martingale, we always mean that it is a martingale relative to $\mathcal{F}$. When we say that a one-parameter process, derived from a multiparameter process by following along a switching strategy $T(t)$, is a martingale, we mean that it is a martingale relative to $\mathcal{F}^T$.

A real-valued function defined on D will be called *multiconcave*, if it is concave in each component separately. It is *multilinear*, if it is linear in each component separately.

Proposition 3. (1) *If* $\mathcal{M}_s = (\mathcal{M}^1_{s_1}, \ldots, \mathcal{M}^d_{s_d})$ *is a multiparameter (super)martingale and* $T(t)$ *is a switching strategy, then* $\mathcal{M}_{T(t)}$ *is a (super)martingale.*

(2) *If* w *is multilinear, then* $w(X^T(t))$ *is a martingale for any strategy* $T(t)$.

(3) *If* w *is multiconcave, then* $w(X^T(t))$ *is a supermartingale for any strategy* $T(t)$.

For $d = 2$, these results follow from Propositions 2.4 and 3.1 in [5]. The proofs given in [5] apply to $d > 2$ as well.

For the edge-data problem, let E denote the set of edges of D and for the facet-data problem, let E denote the set of facets of D. Then, in either case, τ is the first hitting time of E.

Proposition 4. *Let* w *be a continuous, multiconcave function on* D *that agrees with* f *on* E. *If there exists a switching strategy* $\tilde{T}(t)$ *such that* $w(B_{\tilde{T}(t\wedge\tau)})$ *is a martingale, then* w *is the value function* v *defined in* (10) *and* $\tilde{T}(t)$ *is an optimal switching strategy.*

Proof. Appealing to (3) of Proposition 3 and the optional sampling theorem, we conclude that

$$w(x) \geq \mathbf{E}_x w(B_{T(\tau)}) = \mathbf{E}_x f(B_{T(\tau)}) \tag{15}$$

for any switching strategy $T(t)$. Since $w(B_{\tilde{T}(t)})$ is a martingale we see that

$$w(x) = \mathbf{E}_x w(B_{\tilde{T}(\tau)}) = \mathbf{E}_x f(B_{\tilde{T}(\tau)}). \tag{16}$$

From (15) and (16), we conclude that w is the value function and that $\tilde{T}(t)$ is an optimal switching strategy. □

Now, to prove Theorems 1 and 2, two tasks remain:

- Exhibit a function w that is continuous, multiconcave and agrees with f on E.
- Describe a switching strategy $\tilde{T}(t)$ for which $w(B_{\tilde{T}(t)})$ is a martingale.

In order for $w(B_{\tilde{T}(t)})$ to be a martingale it seems to be necessary that $w(x)$ be linear in at least one component at every point $x \in D \setminus E$. Hence, the function w

should be a solution to the following nonlinear Dirichlet problem:

$$\max_{i:0<x_i<a_i} \frac{\partial^2 w}{\partial x_i^2}(x) = 0 \qquad \text{for } x \in D \setminus E \tag{17}$$

$$w(x) = f(x) \qquad \text{for } x \in E. \tag{18}$$

In the next two sections, we construct twice continuously differentiable solutions to this differential equation.

Let w denote the solution to (17), (18) corresponding to either the "edge-data" problem or the "face-data" problem. We finish this section by constructing a switching strategy $\tilde{T}(t)$ for which $w(B_{\tilde{T}(t)})$ is a martingale. Consider any switching strategy $T(t)$. Since the functions x_i, and $x_i x_j$, for $j \neq i$, are multilinear, it follows from part 2 of Proposition 3 that

$$X_i^T(t) \text{ and } X_i^T(t) X_j^T(t)$$

are martingales. Hence, for $t \geq 0$, the quadratic covariation between $X_i^T(t)$ and $X_j^T(t)$ vanishes:

$$\langle X_i^T, X_j^T \rangle_t = 0. \tag{19}$$

For each $i = 1, \ldots, d$, the multiparameter process $(B_{s_i}^i)^2 - s_i$ is a multiparameter martingale and so, by part 1 of Proposition 3, we see that

$$(X_i^T(t))^2 - T_i(t)$$

is a martingale. Hence, for $t \geq 0$, the quadratic variation of $X_i^T(t)$ is given by

$$\langle X_i^T \rangle_t = T_i(t). \tag{20}$$

Since the function w constructed in either of the next two sections is C^2, we can apply Ito's formula together with (19) and (20) to get that

$$\begin{aligned} w(X^T(t)) - w(X^T(0)) &= \sum_i \int_0^t \frac{\partial w}{\partial x_i}(X^T(s)) dX_i^T(s) \\ &\quad + \sum_i \int_0^t \frac{\partial^2 w}{\partial x_i^2}(X^T(s)) dT_i(s). \end{aligned} \tag{21}$$

Theorem 12 in [3] establishes the existence of a strategy $\tilde{T}(t)$ that "follows the smallest index function":

$$\tilde{T}_i(t) \text{ increases only when } X^{\tilde{T}}(t) \in N_i. \tag{22}$$

(In addition to existence, [3] also proves that the strategy is unique if the index processes $\Gamma_i(B_{s_i}^i)$ are simultaneously flat with probability zero. This condition is certainly met here for any pair of index processes and hence for any collection of them.) Whether considering the "edge-data" problem or the "facet-data" problem, in either case, the solution of (17), (18) constructed in the following sections has the property that

$$\frac{\partial^2 w}{\partial x_i^2}(x) = 0 \text{ for } x \in N_i \tag{23}$$

(this follows from the fact that $N_i \subset M_i^c$). Combining (22) and (23), we see that the second sum in (21) vanishes and so $w(X^{\tilde{T}}(t))$ is a martingale.

3. Edge-Data.

Let $w(x)$ denote a candidate for the value function $v(x)$ for the edge-data problem. Assuming that Theorem 1 correctly describes the optimal control regions, we see that $w(x)$ in the control region M_j should be linear in every component except perhaps the j^{th}. That is, the restriction of w to the intersection of the plane determined by a level set of x_j and the control set M_j should be multilinear. Hence, we can "sweep out" $w(x)$ to the boundary of M_j. For $d = 2$ and for $x \in M_1$ this means that we can write:

$$w(x_1, x_2) = (1 - \frac{x_2}{\xi_2(\Gamma_1(x_1))})w(x_1, \xi_2(\Gamma_1(x_1))) + \frac{x_2}{\xi_2(\Gamma_1(x_1))}w(x_1, 0),$$

where

$$\xi_i(u) = \Gamma_i^{-1}(u \wedge \bar{u}_i), \tag{24}$$

and

$$\bar{u}_i = \Gamma_i(a_i). \tag{25}$$

Similarly, for $d = 3$ and $x \in M_1$, the formula becomes

$$\begin{aligned} w(x_1, x_2, x_3) \quad = \quad & (1 - \frac{x_2}{\xi_2(\Gamma_1(x_1))})(1 - \frac{x_3}{\xi_3(\Gamma_1(x_1))})w(x_1, 0, 0) \\ & + \frac{x_2}{\xi_2(\Gamma_1(x_1))}(1 - \frac{x_3}{\xi_3(\Gamma_1(x_1))})w(x_1, \xi_2(\Gamma_1(x_1)), 0) \\ & + (1 - \frac{x_2}{\xi_2(\Gamma_1(x_1))})\frac{x_3}{\xi_3(\Gamma_1(x_1))}w(x_1, 0, \xi_3(\Gamma_1(x_1))) \\ & + \frac{x_2}{\xi_2(\Gamma_1(x_1))}\frac{x_3}{\xi_3(\Gamma_1(x_1))}w(x_1, \xi_2(\Gamma_1(x_1)), \xi_3(\Gamma_1(x_1))). \end{aligned}$$

For the general formula, the notation can be streamlined by observing that the x_1 in the argument list for w can be written as $\xi_1(\Gamma_1(x_1))$. In general, for $x \in M_j$,

$w(x_1, \ldots, x_d)$ can be written as a sum over those subsets A of the set of indices $\{1, 2, \ldots, d\}$ that contain j:

$$w(x_1, \ldots, x_d) = \sum_{A:j \in A} \prod_{i \notin A} \left(1 - \frac{x_i}{\xi_i(\Gamma_j(x_j))}\right) \prod_{i \in A, i \neq j} \frac{x_i}{\xi_i(\Gamma_j(x_j))} w(T_A x_j), \tag{26}$$

where $T_A x_j$ is the d-dimensional point whose coordinates are given by

$$(T_A x_j)_i = \begin{cases} \xi_i(\Gamma_j(x_j)) & i \in A \\ 0 & i \notin A. \end{cases} \tag{27}$$

Note that the product over $i \in A, i \neq j$ in (26) can actually be taken over all $i \in A$ since for $i = j$, the factor $x_i/\xi_i(\Gamma_j(x_j))$ is just one.

Notations are greatly simplified if we change coordinates so that $x_i = \xi_i(u_i)$, for each i. Then the function w becomes

$$\tilde{w}(u_1, \ldots, u_d) = w(\xi_1(u_1), \ldots, \xi_d(u_d)), \tag{28}$$

the domain D becomes $\tilde{D} = \{(u_1, \ldots, u_d) : u_i \leq \bar{u}_i \ \forall i\}$ and, for each i, the control region M_i becomes

$$\tilde{M}_i = \{(u_1, \ldots, u_d) \in \tilde{D} : u_i = \max_j u_j\}. \tag{29}$$

Let $\tilde{w}_j$ denote the restriction of $\tilde{w}$ to $\tilde{M}_j$. Then,

$$\tilde{w}_j(u_1, \ldots, u_d) = \sum_{A:j \in A} \prod_{i \notin A} q_i(u_i, u_j) \prod_{i \in A} \xi_i(u_i) \theta_A(u_j), \tag{30}$$

where

$$q_i(u_i, u_j) = 1 - \frac{\xi_i(u_i)}{\xi_i(u_j)}, \tag{31}$$

$$\theta_A(u) = \frac{\tilde{w}(\tilde{T}_A u)}{\prod_{i \in A} \xi_i(u)}, \tag{32}$$

and

$$(\tilde{T}_A u)_i = \begin{cases} \bar{u}_i \wedge u & i \in A \\ 0 & i \notin A. \end{cases} \tag{33}$$

As long as we ensure that $\tilde{w}_j$ and $\tilde{w}_k$ patch together smoothly along their border $\tilde{M}_j \cap \tilde{M}_k$ then it follows that $\tilde{w}$ will be smooth throughout $\tilde{D}$. Since the change of variables (28) involves twice continuously differentiable functions (which follows from our assumption that the functions γ_i are C^3), the smoothness of $\tilde{w}$ in $\tilde{D}$ translates back into the same smoothness of w in D (up to second order).

As we shall now show, stipulating first order smoothness across $\tilde{M}_j \cap \tilde{M}_k$ will force θ_A to be a specific function for each A. Stipulating second order smoothness will force the functions Γ_i to be as defined in (11).

Fix j,k with $j \neq k$. First, we note that the values of $\tilde{w}_j$ and $\tilde{w}_k$ agree along $\tilde{M}_j \cap \tilde{M}_k$:

$$\tilde{w}_j|_{u_j:=u_k:=u} = \sum_{A:j,k\in A} Q_A \Xi_A \theta_A(u) = \tilde{w}_k|_{u_j:=u_k:=u},$$

where Q_A and Ξ_A are abbreviations for the following expressions

$$Q_A = \prod_{i\notin A} q_i(u_i,u) \tag{34}$$

$$\Xi_A = \prod_{i\in A} \xi_i(u_i), \tag{35}$$

and $\tilde{w}_j|_{u_j:=u_k:=u}$ denotes the function $\tilde{w}_j(u_1,\ldots,u_d)$ evaluated at $u_j = u$ and $u_k = u$. It is easy to check that

$$\begin{aligned}\left.\frac{\partial \tilde{w}_j}{\partial u_k}\right|_{u_j:=u_k:=u} &= -\sum_{A:j\in A,k\notin A} \frac{\xi_k'(u)}{\xi_k(u)} Q_{A\cup k}\Xi_A\theta_A(u) \\ &\quad + \sum_{A:j,k\in A} \xi_k'(u) Q_A \Xi_{A\setminus k}\theta_A(u)\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial \tilde{w}_k}{\partial u_k}\right|_{u_j:=u_k:=u} &= \sum_{A:j\notin A,k\in A} \frac{\xi_j'(u)}{\xi_j(u)} Q_{A\cup j}\Xi_A\theta_A(u) \\ &\quad + \sum_{A:j,k\in A}\sum_{i\notin A} \frac{\xi_i(u_i)\xi_i'(u)}{\xi_i^2(u)} Q_{A\cup i}\Xi_A\theta_A(u) \\ &\quad + \sum_{A:j,k\in A} \xi_k'(u) Q_A \Xi_{A\setminus k}\theta_A(u) \\ &\quad + \sum_{A:j,k\in A} Q_A\Xi_A\theta_A'(u).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{\prod_i \xi_i(u_i)}\left.\left(\frac{\partial \tilde{w}_k}{\partial u_k} - \frac{\partial \tilde{w}_j}{\partial u_k}\right)\right|_{u_j:=u_k:=u} &= \sum_{A:j,k\in A} R_A\theta_A'(u) \\ &\quad + \sum_{A:j,k\in A}\sum_{i\notin A} \frac{\xi_i'(u)}{\xi_i^2(u)} R_{A\cup i}\theta_A(u) \\ &\quad + \sum_{A:j\notin A,k\in A} \frac{\xi_j'(u)}{\xi_j^2(u)} R_{A\cup j}\theta_A(u) \\ &\quad + \sum_{A:j\in A,k\notin A} \frac{\xi_k'(u)}{\xi_k^2(u)} R_{A\cup k}\theta_A(u),\end{aligned}$$

where R_A is an abbreviation for the following expression

$$R_A = \frac{Q_A}{\prod_{i\notin A}\xi_i(u_i)} = \prod_{i\notin A}\left(\frac{1}{\xi_i(u_i)} - \frac{1}{\xi_i(u)}\right).$$

Combining the last three sums, we get

$$\frac{1}{\prod_i \xi_i(u_i)} \left(\frac{\partial \tilde{w}_k}{\partial u_k} - \frac{\partial \tilde{w}_j}{\partial u_k} \right) \Bigg|_{u_j:=u_k:=u} = \sum_{A:j,k\in A} R_A \left(\theta'_A(u) + \sum_{i\in A} \frac{\xi'_i(u)}{\xi_i^2(u)} \theta_{A\setminus i}(u) \right).$$

Hence, to guarantee that first derivatives of $\tilde{w}$ are continuous across $\tilde{M}_j \cap \tilde{M}_k$ for all $j \neq k$, it suffices to define θ_A so that

$$\theta'_A = -\sum_{i\in A} \frac{\xi'_i}{\xi_i^2} \theta_{A\setminus i}, \tag{36}$$

for all A containing two or more elements. If we let

$$\bar{u}_A = \min_{j\in A} \bar{u}_j,$$

then $\tilde{T}_A \bar{u}_A$ lies on one of the "back faces" of $\tilde{D}$ (i.e. one of the components is at its upper bound) and so

$$\theta_A(\bar{u}_A) = 0. \tag{37}$$

Also, if A contains exactly one element, say j, then we see from (28), (32), (33), and the fact that w is to agree with γ_j on the j^{th} coordinate axis, that

$$\theta_j(u) = \frac{\gamma_j(\xi_j(u))}{\xi_j(u)}. \tag{38}$$

Hence, starting with sets A of cardinality two and working up, each θ_A is uniquely determined by (36) and (37). Carrying out this recursion we get

$$\theta_A(u) = \sum_{j\in A} \int_{\mathcal{R}_{A\setminus j}(u)} \left(\prod_{i\in A\setminus j} \frac{\xi'_i(u_i)}{\xi_i^2(u_i)} \right) \theta_j(\max_{i\in A\setminus j} u_i) \prod_{i\in A\setminus j} du_i,$$

where

$$\mathcal{R}_A(u) = \{(u_i)_{i\in A} : \ u < u_i \leq \bar{u}_i\}.$$

Finally, we need to check second derivatives. Carefully differentiating, we see that

$$
\begin{aligned}
\frac{\xi_k^2(u)}{\xi_k'(u)} \frac{1}{\prod_i \xi_i(u_i)} \frac{\partial^2 \tilde{w}_j}{\partial u_j \partial u_k}\Big|_{u_j:=u_k:=u} = & \sum_{A:j,k\in A} \sum_{i\notin A} \xi_k(u) \frac{\xi_i'(u)}{\xi_i^2(u)} R_{A\cup i}\theta_A(u) \\
& + \sum_{A:j,k\in A} \xi_k(u) \frac{\xi_j'(u)}{\xi_j(u)} R_A \theta_A(u) \\
& + \sum_{A:j,k\in A} \xi_k(u) R_A \theta_A'(u) \\
& + \sum_{A:j\in A, k\notin A} \frac{\xi_k'(u)}{\xi_k(u)} R_{A\cup k}\theta_A(u) \\
& - \sum_{A:j\in A, k\notin A} \sum_{i\notin A, i\neq k} \frac{\xi_i'(u)}{\xi_i^2(u)} R_{A\cup i\cup k}\theta_A(u) \\
& - \sum_{A:j\in A, k\notin A} \frac{\xi_j'(u)}{\xi_j(u)} R_{A\cup k}\theta_A(u) \\
& - \sum_{A:j\in A, k\notin A} R_{A\cup k}\theta_A'(u).
\end{aligned}
$$

Now substituting (36) into the above formula and reindexing so the θ's always are subscripted with an A, we get

$$
\begin{aligned}
\frac{\xi_j^2(u)}{\xi_j'(u)} \frac{\xi_k^2(u)}{\xi_k'(u)} \frac{1}{\prod_i \xi_i(u_i)} \frac{\partial^2 \tilde{w}_j}{\partial u_j \partial u_k}\Bigg|_{u_j:=u_k:=u} = & - \sum_{A:j\notin A, k\in A} \xi_k(u) R_{A\cup j}\theta_A(u) \\
& - \sum_{A:j\in A, k\notin A} \xi_j(u) R_{A\cup k}\theta_A(u) \\
& + \sum_{A:j,k\in A} \xi_j(u)\xi_k(u) R_A\theta_A(u) \\
& + \sum_{A:j,k\notin A, |A|\geq 1} R_{A\cup j\cup k}\theta_A(u) \\
& - R_{j\cup k}\theta_j'(u) \frac{\xi_j^2(u)}{\xi_j'(u)}.
\end{aligned}
$$

Interchanging the roles of j and k, we can write down the analogous expression for $\tilde{w}_k$ and then subtract to get

(39)

$$
\frac{\xi_j^2(u)}{\xi_j'(u)} \frac{\xi_k^2(u)}{\xi_k'(u)} \frac{1}{\prod_i \xi_i(u_i)} \left(\frac{\partial^2 \tilde{w}_j}{\partial u_j \partial u_k} - \frac{\partial^2 \tilde{w}_k}{\partial u_j \partial u_k} \right)\Bigg|_{u_j:=u_k:=u} = R_{j\cup k} \left(\theta_k'(u) \frac{\xi_k^2(u)}{\xi_k'(u)} - \theta_j'(u) \frac{\xi_j^2(u)}{\xi_j'(u)} \right).
$$

Now we are almost home. Recalling (38), we see that

$$
\theta_k(u) = \frac{\gamma_k(\xi_k(u))}{\xi_k(u)}
$$

and so, using (11) and suppressing the dependent variable u, we get

$$\theta_k' \frac{\xi_k^2}{\xi_k'} = \xi_k \gamma_k'(\xi_k) - \gamma_k(\xi_k) = -\Gamma_k(\xi_k) = -u.$$

Hence, both sides of the difference on the right-hand side of (39) are equal to $-u$ and so the difference vanishes.

4. Face-Data.

Let $w(x)$ denote a candidate for the value function $v(x)$ for the face-data problem. As in the previous section, it is convenient to work in the system of coordinates defined by (28). Hence, the control region N_i described in Theorem 2 becomes $\tilde{N}_i = \{(u_1, \ldots, u_d) : u_i = \min_j u_j\}$.

Assuming that Theorem 2 correctly describes the optimal control regions, we see that $w(x)$ should be linear in x_i on the set N_i. Hence, we can "sweep out" $w(x)$ to the boundary of N_i. Using the u_i coordinates, this sweeping becomes

$$\tilde{w}(u_1, \ldots, u_d) = \left(1 - \frac{\xi_i(u_i)}{\xi_i(u_j)}\right) \prod_{k \neq i} \gamma_k(u_k) + \frac{\xi_i(u_i)}{\xi_i(u_j)}\ \tilde{w}(u_1, \ldots, u_d)|_{u_i ::= u_j}\,, \tag{40}$$

for

$$(u_1, \ldots, u_d) \in \tilde{N}_{i,j} = \{(u_1, \ldots, u_d) \in \tilde{D} : u_i \le u_j \le \min_{k \neq i,j} u_k\}.$$

First, we make sure that $\tilde{w}$ is twice continuously differentiable across the boundary between $\tilde{N}_{i,j}$ and $\tilde{N}_{j,i}$. Let $\tilde{w}_{i,j}$ denote the restriction of $\tilde{w}$ to $\tilde{N}_{i,j}$ so that $\tilde{w}_{i,j}$ is given by the right-hand side in (40). From (40), we see that

$$\left.\frac{\partial \tilde{w}_{i,j}}{\partial u_i}\right|_{u_i := u_j := u} = -\frac{\xi_i'(u)}{\xi_i(u)} \gamma_j(u) \prod_{k \neq i,j} \tilde{\gamma}_k(u_k) + \frac{\xi_i'(u)}{\xi_i(u)}\ \tilde{w}_{i,j}|_{u_i := u_j := u}$$

and

$$\left.\frac{\partial \tilde{w}_{j,i}}{\partial u_i}\right|_{u_i := u_j := u} = \frac{\xi_j'(u)}{\xi_j(u)} \gamma_i(u) \prod_{k \neq i,j} \tilde{\gamma}_k(u_k) - \frac{\xi_j'(u)}{\xi_j(u)}\ \tilde{w}_{j,i}|_{u_i := u_j := u} + \frac{\partial}{\partial u}\ \tilde{w}_{j,i}|_{u_i := u_j := u}\,,$$

where $\tilde{\gamma}_k$ is defined by

$$\tilde{\gamma}_k(u_k) = \gamma_k(\xi_k(u_k))).$$

Hence, for first derivatives to match we need

$$\begin{aligned} \frac{\partial}{\partial u}\ \tilde{w}_{j,i}|_{u_i := u_j := u} \quad &- \quad \left(\frac{\xi_i'(u)}{\xi_i(u)} + \frac{\xi_j'(u)}{\xi_j(u)}\right) \tilde{w}_{j,i}|_{u_i := u_j := u} \\ &= \quad -\left(\frac{\xi_i'(u)}{\xi_i(u)} \gamma_j(u) + \frac{\xi_j'(u)}{\xi_j(u)} \gamma_i(u)\right) \prod_{k \neq i,j} \tilde{\gamma}_k(u_k). \end{aligned} \tag{41}$$

At this point let us consider that part of the state space where $u_1 \le u_2 \le \cdots \le u_d$. From (40) and (41), we see that

$$\tilde{w}(u_1,\ldots,u_d) = \left(1 - \frac{\xi_1(u_1)}{\xi_1(u_2)}\right) G_1 + \frac{\xi_1(u_1)}{\xi_1(u_2)}\theta_2(u_2), \tag{42}$$

where G_i is an abbreviation for

$$G_i = \prod_{k>i} \tilde{\gamma}_k(u_k)$$

and $\theta_2(u) = \tilde{w}(u,u,u_3,\ldots,u_d)$ is a solution of

$$\theta_2' - \left(\frac{\xi_1'}{\xi_1} + \frac{\xi_2'}{\xi_2}\right)\theta_2 = -G_2\left(\frac{\xi_1'}{\xi_1}\gamma_2 + \frac{\xi_2'}{\xi_2}\gamma_1\right).$$

Using the integrating factor $1/\xi_1\xi_2$, we can solve for θ_2:

$$\theta_2(u) = \xi_1(u)\xi_2(u)\left(\frac{\theta_2(u_3)}{\xi_1(u_3)\xi_2(u_3)} + G_2\int_u^{u_3}\left(\frac{\xi_1'\tilde{\gamma}_2}{\xi_1^2\xi_2} + \frac{\xi_2'\tilde{\gamma}_1}{\xi_2^2\xi_1}\right)\right),$$

where we have suppressed the integration variable and its differential in the above integral. To keep notations in check, we will suppress integration variables and differentials several times in the following expressions. We hope this adds to the clarity of the formulas. Substituting this formula for θ_2 into (42), we get

$$\begin{aligned} \tilde{w}(u_1,\ldots,u_d) &= \Xi_1 G_1 \int_{u_1}^{u_2}\frac{\xi_1'}{\xi_1^2} \\ &\quad + \Xi_2 G_2 \int_{u_2}^{u_3}\left(\frac{\xi_1'\tilde{\gamma}_2}{\xi_1^2\xi_2} + \frac{\xi_2'\tilde{\gamma}_1}{\xi_2^2\xi_1}\right) \\ &\quad + \frac{\Xi_2}{\xi_1(u_3)\xi_2(u_3)}\theta_3(u_3), \end{aligned} \tag{43}$$

where Ξ_i is an abbreviation for

$$\Xi_i = \prod_{k\le i}\xi_k(u_k),$$

$$\theta_3(u) = \tilde{w}(u,u,u,u_4,\ldots,u_d),$$

and the first term from (42) has been rewritten using the following simple identity

$$\left(1 - \frac{\xi_1(u_1)}{\xi_1(u_2)}\right) = \xi_1(u_1)\int_{u_1}^{u_2}\frac{\xi_1'}{\xi_1^2}.$$

Equation (41) can be used to get a directional derivative of $\tilde{w}$ at a point of the form $(u,u,u,u_4,\ldots,u_d)$. Writing down the analogous expressions obtained by considering the cases $u_2 \ge \max(u_1,u_3)$ and $u_1 \ge \max(u_2,u_3)$, we can get two more independent directional derivatives. From these three independent directional derivatives, it is not hard to see that θ_3 satisfies the following differential equation

$$\theta_3' - \left(\frac{\xi_1'}{\xi_1} + \frac{\xi_2'}{\xi_2} + \frac{\xi_3'}{\xi_3}\right)\theta_3 = -\left(\frac{\xi_1'}{\xi_1}\tilde{\gamma}_2\tilde{\gamma}_3 + \frac{\xi_2'}{\xi_2}\tilde{\gamma}_1\tilde{\gamma}_3 + \frac{\xi_3'}{\xi_3}\tilde{\gamma}_1\tilde{\gamma}_2\right).$$

The integrating factor for this differential equation is $1/\xi_1\xi_2\xi_3$. Integrating to solve for θ_3 and substituting into (43), we get

$$\begin{aligned}\tilde{w}(u_1,\ldots,u_d) \quad &= \quad \Xi_1 G_1 \int_{u_1}^{u_2} \frac{\xi_1'}{\xi_1^2} \\ &+\Xi_2 G_2 \int_{u_2}^{u_3} \left(\frac{\xi_1'\tilde{\gamma}_2}{\xi_1^2\xi_2} + \frac{\xi_2'\tilde{\gamma}_1}{\xi_2^2\xi_1} \right) \\ &+\Xi_3 G_3 \int_{u_3}^{u_4} \left(\frac{\xi_1'\tilde{\gamma}_2\tilde{\gamma}_3}{\xi_1^2\xi_2\xi_3} + \frac{\xi_2'\tilde{\gamma}_1\tilde{\gamma}_3}{\xi_1\xi_2^2\xi_3} + \frac{\xi_3'\tilde{\gamma}_1\tilde{\gamma}_2}{\xi_1\xi_2\xi_3^2} \right) \\ &+\frac{\Xi_3}{\xi_1(u_4)\xi_2(u_4)\xi_3(u_4)}\theta_4(u_4),\end{aligned}$$

where

$$\theta_4(u) = \tilde{w}(u,u,u,u,u_5,\ldots,u_d).$$

Now it is easy to see how this process must continue. Ultimately, we get

$$\tilde{w}(u_1,\ldots,u_d) = \sum_{i=1}^{d} \Xi_i G_i \int_{u_i}^{u_{i+1}} \sum_{k=1}^{i} \frac{\xi_k'}{\xi_k\tilde{\gamma}_k} \prod_{j\le i} \frac{\tilde{\gamma}_j}{\xi_j},$$

where we have put $u_{d+1} = \bar{u}$ and used the fact that $\tilde{w}$ vanishes at $(\bar{u},\ldots,\bar{u})$. On the other parts of the state space it is clear that we get an analogous formula with the indices changed so that an index i is replaced with the index of the i^{th} smallest u value.

Now $\tilde{w}$ is defined everywhere in $\tilde{D}$ and is continuously differentiable throughout. It remains to show that it is also twice continuously differentiable. It is sufficient to show that second derivatives agree across the boundary between $\tilde{N}_{i,j}$ and $\tilde{N}_{j,i}$. Straightforward calculation shows that

(44)
$$\frac{\xi_i\xi_j}{\xi_i'\xi_j'} \left(\frac{\partial^2 \tilde{w}_{i,j}}{\partial u_i \partial u_j} - \frac{\partial^2 \tilde{w}_{j,i}}{\partial u_i \partial u_j} \right)\Bigg|_{u_i:=u_j:=u} = \left(\frac{\xi_i(u)}{\xi_i'(u)}\tilde{\gamma}_i'(u) - \tilde{\gamma}_i(u) - \frac{\xi_j(u)}{\xi_j'(u)}\tilde{\gamma}_j'(u) + \tilde{\gamma}_j(u) \right) \prod_{k\ne i,j} \tilde{\gamma}_k.$$

Since $\tilde{\gamma}_i(u) = \gamma_i(\xi_i(u))$, we see that by (11)

$$\frac{\xi_i(u)}{\xi_i'(u)}\tilde{\gamma}_i'(u) - \tilde{\gamma}_i(u) = \xi_i(u)\gamma_i'(\xi_i(u)) - \gamma_i(\xi_i(u)) = -\Gamma_i(\xi_i(u)) = -u.$$

Similarly, the last two terms in parentheses in (44) add up to u and so the right-hand side vanishes. This completes the proof that w is twice continuously differentiable in D.

5. Other Examples.

Perhaps the most natural extension of the preceding results would be to consider more general biconcave data on the d faces adjacent to the origin. It seems that such generalizations will be quite difficult. Indeed, for $d = 3$ we considered the following data:

$$f(x) = \begin{cases} \gamma_0(x_2)\gamma_1(x_3) & \text{if } x = (0, x_2, x_3) \\ \gamma_0(x_3)\gamma_1(x_1) & \text{if } x = (x_1, 0, x_3) \\ \gamma_0(x_1)\gamma_1(x_2) & \text{if } x = (x_1, x_2, 0) \\ 0 & \text{otherwise} \end{cases},$$

where γ_0 and γ_1 are two different strongly concave functions which vanish at the endpoints of their domains. By considering a discretization of the Dirichlet problem (17), (18), we were able to apply the method of successive approximations to numerically solve for the value function and hence the optimal strategy. As in Figure 2, there are switching surfaces emanating from the three coordinate axes, but this time they do not meet along a single curve. In fact, the behavior of the optimal switching strategy is quite intricate inside a triangular tube enclosed by the three surfaces. It would be very interesting to understand more about the nature of examples such as this one.

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