

WHY SUNSET REFLECTIONS ON WATER SMEAR VERTICALLY

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ABSTRACT

Everyone has seen sunsets and/or sunrises over water such as a lake or an ocean. We have all noticed that the fact that the water is not perfectly smooth causes the reflection of the sun to smear. And, I'm sure that most of us have noticed that the smear is always vertical. The natural question to ask is: why vertical? In this paper, we will show that the size of the angular smear in the vertical direction is larger than in the horizontal direction by a factor of $1/\theta$, where θ is the angular height of the sun above the horizon measured in radians.

1. INTRODUCTION

Reflections on water are almost always smeared out a bit because water in an outdoor setting tends not to be a perfectly smooth mirror. There is almost always some random oscillation (aka chop) of the surface. Figure 1 shows some photographs illustrating the effect. A simple explanation for why the smearing is vertical is that most sunset pictures are taken by someone standing at the shore looking straight out perpendicular to the shore and hence the waves are flowing directly toward the photographer. It turns out that this simple answer is incorrect. The smear is vertical even when the waves are not flowing toward the photographer.

In this paper we derive an explicit formula for the magnitude of the angular displacement of the reflection for different angular deviations of the water surface from horizontally flat. We will show that left/right tilts of the surface of the water produce much smaller reflective displacements than to and fro rotations.

The anisotropic smearing question has been raised by others. For example, it was raised by a high school student on the *Physics Stack Exchange* website [jack](https://physics.stackexchange.com/users/30212/jack) (<https://physics.stackexchange.com/users/30212/jack>). The question inspired some responses but, to the student's frustration, none of them explained why the smearing is stronger in the vertical direction.

There is, of course, an extensive literature on how light reflects off from rough surfaces such as an ocean or a lake. The earliest paper analyzing reflections on water is [Minnaert \(1942\)](#). The classic references in this area are [Minnaert \(1954\)](#) and [Cox and Munk \(1954\)](#). A more recent republication of the Minnaert book is [Minnaert \(1995\)](#). It includes a derivation of a simple explicit formula that makes it clear why low-angle reflections on water smear more vertically than they do horizontally.

2. TO AND FRO ANGULAR TILTS

Figure 2 shows the sunbeam arriving to the observer directly and after reflecting off a perfectly flat horizontal water surface. It is clear from the diagram that if the Sun is at an angle θ above the horizon, then the reflection is at an angle θ below the horizon.

Consider the sunbeam hitting the water surface at a place where the surface is tilted a small angle α toward the observer. Figure 3 illustrates this scenario. The sunbeam hits the water surface at an angle $\theta - \alpha$ (measured from tangency not normalcy). The reflected beam departs the surface at this same angle and therefore appears to the observer to be coming from an angle $\theta - 2\alpha$ below the horizon. Since the “unsmeared” reflection came from an angle θ below the horizon, we conclude that the angular smearing associated with this to/fro rotation is 2α .

For α between 0 and $\theta/2$, the smeared reflection appears closer to the horizon than the perfect reflection. For $\alpha = \theta/2$, the reflection appears to emanate from the horizon. For $\alpha < 0$, the smeared reflection appears to come from a place closer to the observer.

Clearly, if α is greater than $\theta/2$, then the reflected beam does not rise above the horizon and just rehits the water

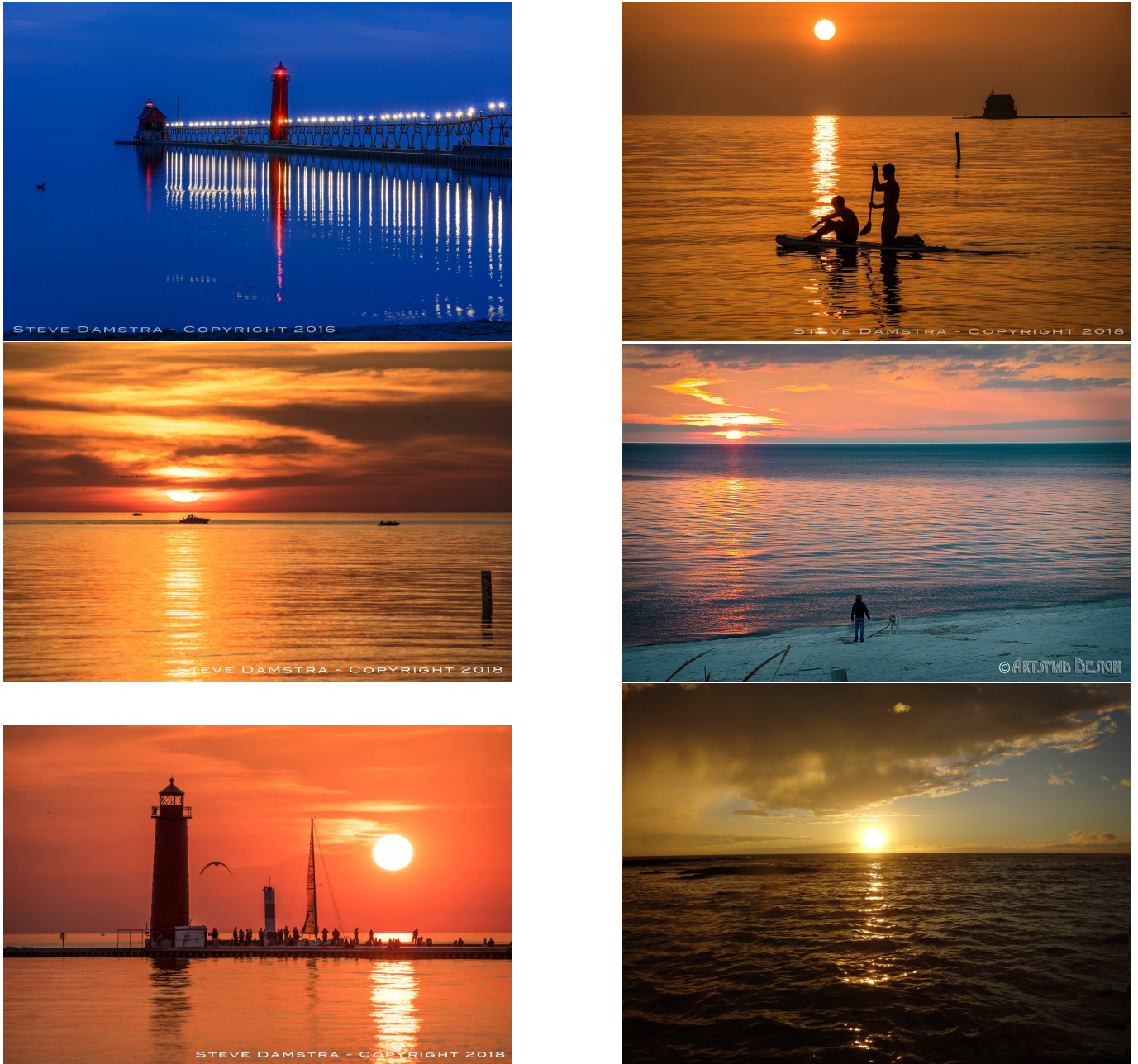


Figure 1. A few sunset (and related) photos. Credit: These pictures were taken by Steve Damstra.

producing a second reflection. These multiple reflections contribute significantly to the “toward the observer” smearing. In this paper, we will focus our attention on the smearing caused by single reflections.

3. LEFT/RIGHT ANGULAR TILTS

For the left/right tiltings of the water surface, we need to work in 3D space. To that end, let us assume that displacements to the left and right are displacements parallel to the x -axis in our coordinate system and that the positive y -axis points from the observer toward the Sun and hence that the z -axis is the vertical axis. The unit vector that defines the direction toward the Sun is

$$\vec{v} = (0, \cos \theta, \sin \theta)$$

and the unit vector that points out from an unsmearred location on the water will be denote by

$$\vec{u} = (0, -\cos \theta, \sin \theta).$$

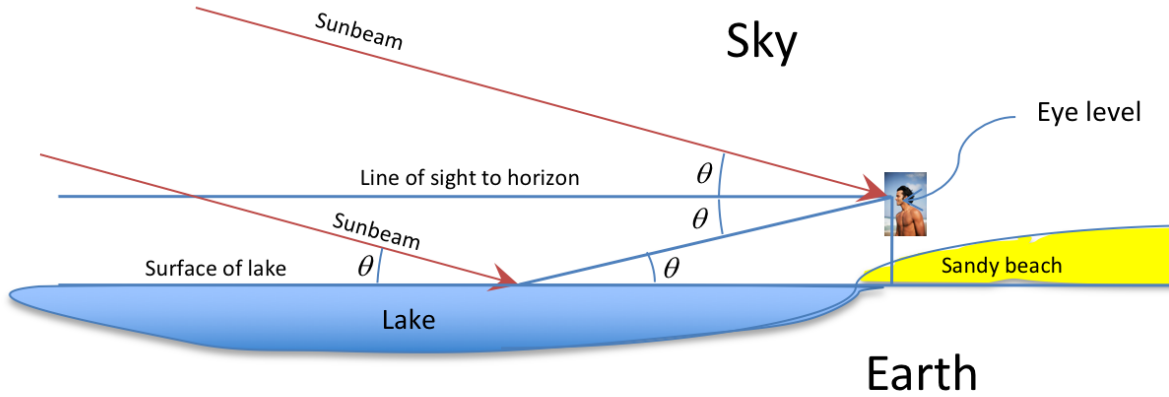


Figure 2. The sunbeam and its reflection when the water is perfectly horizontal.

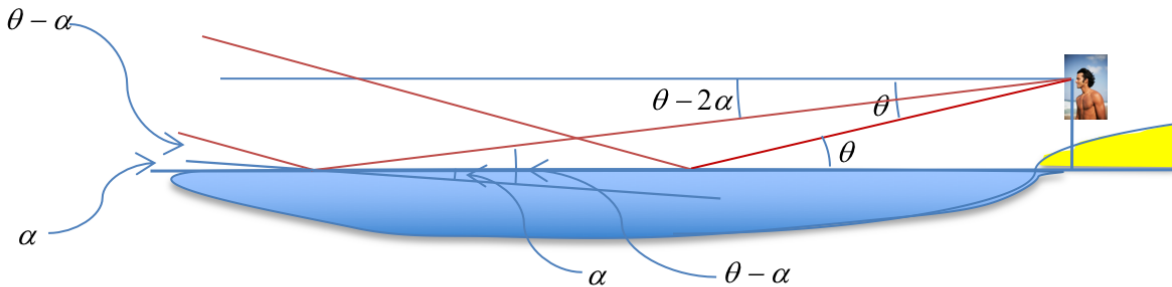


Figure 3. The sunbeam and its reflection when the reflection point is tipped a small angle α toward the observer.

If the water surface is tilted, say, to the right by an angle α , then the normal vector to the water surface at this location is

$$\vec{n} = (\sin \alpha, 0, \cos \alpha).$$

The unit direction vector for the beam that is reflected by this tilted water surface is then given by

$$\begin{aligned} \vec{w} &= (\vec{v} \cdot \vec{n})\vec{n} - (\vec{v} - (\vec{v} \cdot \vec{n})\vec{n}) \\ &= 2(\vec{v} \cdot \vec{n})\vec{n} - \vec{v} \end{aligned}$$

where the dot denotes the inner product of a pair of vectors. It is easy to plug in the definitions of \vec{v} and \vec{n} and do some simple algebra to see that

$$\vec{w} = (\sin \theta \sin(2\alpha), -\cos \theta, \sin \theta \cos(2\alpha)).$$

Of course, this vector must be a unit vector — it is easy to check that it is. Let ψ denote the angle between the direction vector associated with the non-smearing reflection and the smeared reflection. Given that the unit direction vectors for these two rays of light are \vec{u} and \vec{w} , we see that

$$\begin{aligned}\cos \psi &= \vec{u} \cdot \vec{w} = \cos^2 \theta + \sin^2 \theta \cos(2\alpha) \\ &= 1 - 2 \sin^2 \theta \sin^2 \alpha.\end{aligned}$$

Hence, we get that

$$\psi = \cos^{-1} (1 - 2 \sin^2 \theta \sin^2 \alpha).$$

If we assume that the angles θ and α are small, then we can use simple approximations to get that

$$\begin{aligned}1 - 2 \sin^2 \theta \sin^2 \alpha &\approx 1 - 2\theta^2 \alpha^2 \\ &\approx \cos(2\theta\alpha).\end{aligned}$$

And so we get

$$\psi \approx 2\theta\alpha.$$

In conclusion, we see that the left/right smearing differs from the to/fro smearing by a factor θ , where θ is given in radians. So, for example, if the Sun is just 5 degrees above the horizon, then $\theta = 5 \times \frac{\pi}{180} = 0.087$. In other words, the to/fro smearing is more than 10 times greater than the left/right smearing.

4. THE GENERAL CASE

Now suppose that the normal vector is tilted at an angle α from vertical but the direction of the tilt is at some angle β where $\beta = 0$ corresponds to being tilted directly at the observer (as considered in Section 2) and $\beta = \pi/2$ radians corresponds to the tilt being to the right as considered in Section 3.

In this general case, the normal vector is given by

$$\vec{n} = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha).$$

It is easy to check that the unit vector associated with the reflected beam is given by

$$\begin{aligned}\vec{w} &= 2((0, \cos \theta, \sin \theta) \cdot (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha))(\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) - (0, \cos \theta, \sin \theta) \\ &= 2(\cos \theta \sin \alpha \cos \beta + \sin \theta \cos \alpha)(\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) - (0, \cos \theta, \sin \theta) \\ &= (2(\cos \theta \sin \alpha \cos \beta + \sin \theta \cos \alpha) \sin \alpha \sin \beta, \\ &\quad 2(\cos \theta \sin \alpha \cos \beta + \sin \theta \cos \alpha) \sin \alpha \cos \beta - \cos \theta, \\ &\quad 2(\cos \theta \sin \alpha \cos \beta + \sin \theta \cos \alpha) \cos \alpha - \sin \theta).\end{aligned}$$

Multiplying by \vec{u} things simplify a bit:

$$\begin{aligned}\cos \psi &= \vec{u} \cdot \vec{w} = 2(\sin^2 \theta \cos^2 \alpha - \cos^2 \theta \sin^2 \alpha \cos^2 \beta) + \cos(2\theta) \\ &= 1 - 2 \sin^2 \theta \sin^2 \alpha - 2 \cos^2 \theta \sin^2 \alpha \cos^2 \beta \\ &= 1 - 2 \sin^2 \alpha + 2 \cos^2 \theta \sin^2 \alpha \sin^2 \beta \\ &= \cos(2\alpha) + 2 \cos^2 \theta \sin^2 \alpha \sin^2 \beta.\end{aligned}$$

When $\beta = \pi/2$, this formula for \vec{w} reduces to the formula derived in Section 3 and, when $\beta = 0$, this formula reduces to

$$\cos \psi = \vec{u} \cdot \vec{w} = \cos(2\alpha)$$

and hence that $\psi = 2\alpha$ as derived in Section 2.

5. NUMERICAL SIMULATION

A Matlab program to generate pictures of what a reflection should look like in water with some chop but no directional waves. The output of this program is shown in Figure 4.

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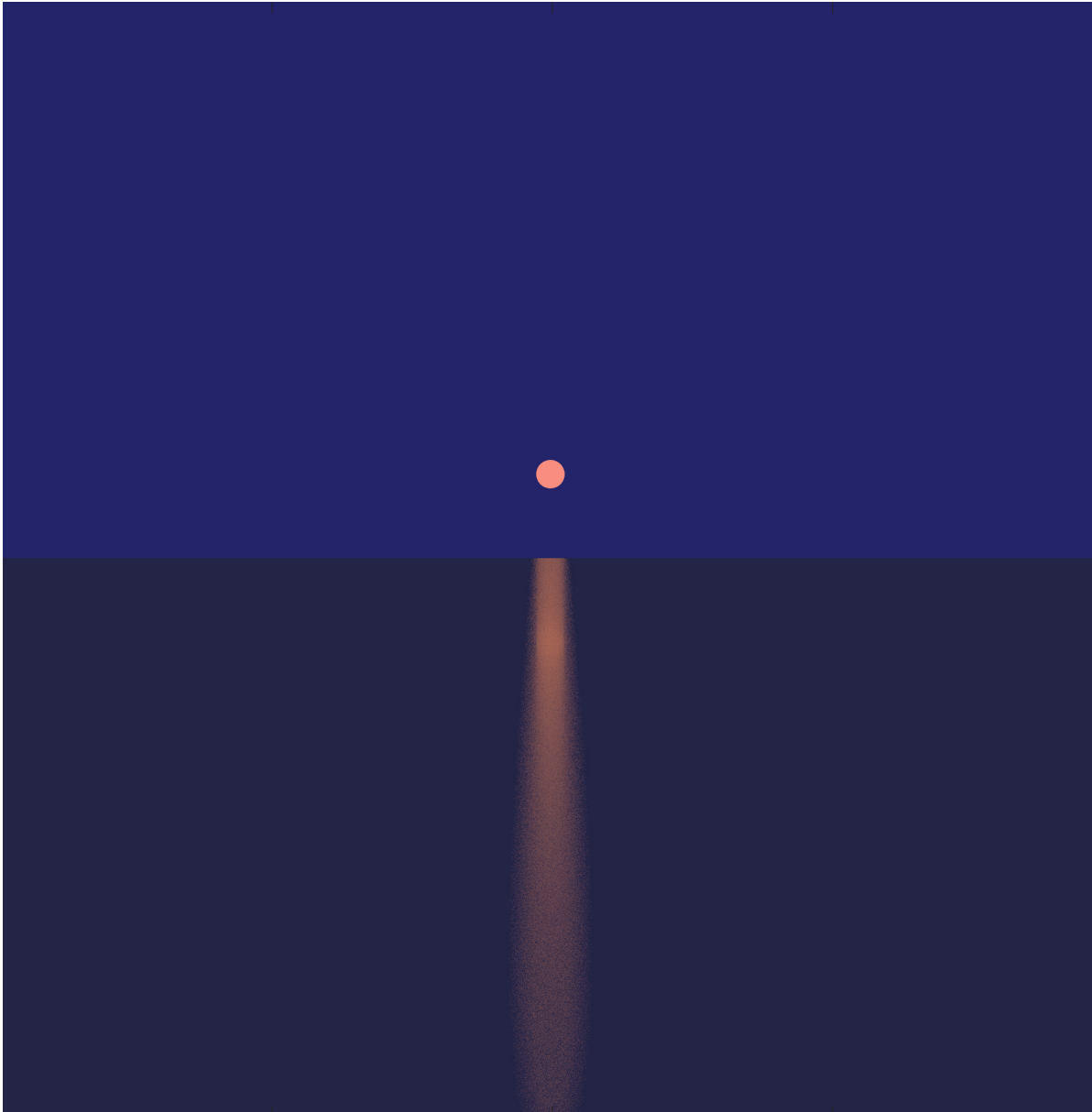


Figure 4. A simulation of the reflection when the Sun is 1.5° above the horizon.

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