

Engineering Applications of Nonlinear Optimization

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ABSTRACT

- Brief description of interior-point methods for NLP.
- Engineering Applications
 - Telescope design
 - N -Body problem
 - Antenna Array design
 - Finite Impulse Response (FIR) filter design

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Interior-Point Methods for NLP



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LOQO: An Interior-Point Code for NLP



LOQO solves problems in the following form:

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && b \leq h(x) \leq b + r, \\ & && l \leq x \leq u \end{aligned}$$

The functions $f(x)$ and $h(x)$ must be twice differentiable (at least at points of evaluation).

The standard *interior-point paradigm* is used:

- Add slacks.
- Replace nonnegativities with barrier terms in objective.
- Write first-order optimality conditions.
- Rewrite optimality conditions in primal-dual symmetric form.
- Use Newton's method to get search directions...

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Interior-Point Paradigm Continued

- Use Newton's method to get search directions:

$$\begin{bmatrix} -H(x, y) - D & A^T(x) \\ A(x) & E \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A^T(x)y \\ -h(x) + \mu Y^{-1}e \end{bmatrix}.$$

Here, D and E are diagonal matrices involving slack variables,

$$H(x, y) = \nabla^2 f(x) - \sum_{i=1}^m y_i \nabla^2 h_i(x) + \lambda I, \text{ and } A(x) = \nabla h(x),$$

where λ is chosen to ensure appropriate descent properties.

- Compute step lengths to ensure positivity of slack variables.
- Shorten steps further to ensure a reduction in either infeasibility or in the barrier function—a myopic, or Markov, *filter*. (N.B.: We no longer use a merit function.)
- Step to new point and repeat.

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Telescope Design

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The Big Question: Are We Alone?

- Are there Earth-like planets?
- Are they common?
- Is there life on some of them?



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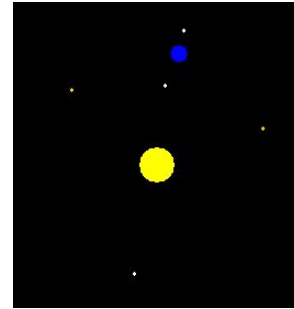
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Exosolar Planets—Where We Are Now

There are more than 120 Exosolar planets known today.

They were discovered by detecting a sinusoidal doppler shift in the parent star's spectrum due to gravitationally induced **wobble**.



This method works best for large Jupiter-sized planets with close-in orbits.

One of these planets, HD209458b, also transits its parent star once every 3.52 days. These transits have been detected photometrically as the star's light flux decreases by about 1.5% during a transit.



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Terrestrial Planet Finder Telescope

- NASA/JPL space telescope.
- Launch date: 2014
- DETECT: Search 150-500 nearby (5-15 pc distant) Sun-like stars for Earth-like planets.
- CHARACTERIZE: Determine basic physical properties and measure “biomarkers”, indicators of life or conditions suitable to support it.



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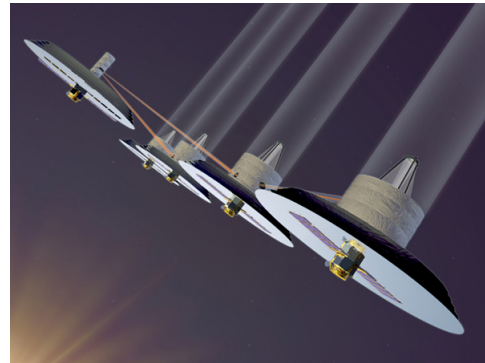
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Why Is It Hard? Can't Hubble do it?

- If the star is Sun-like and the planet is Earth-like, then the reflected visible light from the planet is 10^{-10} times as bright as the star. This is a difference of 25 magnitudes!
- If the star is 10 pc (33 ly) away and the planet is 1 AU from the star, the angular separation is 0.1 arcseconds!

Originally, it was thought that this would require a space-based infrared nulling interferometer (as shown).

However, a more recent idea is to use a single large visible-light telescope with an elliptical mirror (4 m x 10 m) and a *shaped pupil* for diffraction control.



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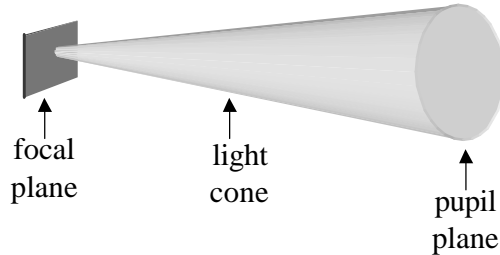
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The Shaped Pupil Concept

Consider a telescope. Light enters the front of the telescope—the *pupil plane*.

The telescope focuses the light passing through the pupil plane from a given direction at a certain point on the *focal plane*, say $(0, 0)$.



However, a point source produces not a point image but an *Airy pattern* consisting of an *Airy disk* surrounded by a system of *diffraction rings*.

These diffraction rings are too bright. An Earth-like planet is only about 10^{-10} times as bright as its Sun-like star. The rings would completely hide the planet.

By placing a mask over the pupil, one can control the shape and strength of the diffraction rings. The problem is to find an optimal shape so as to put a very deep *null* very close to the Airy disk.



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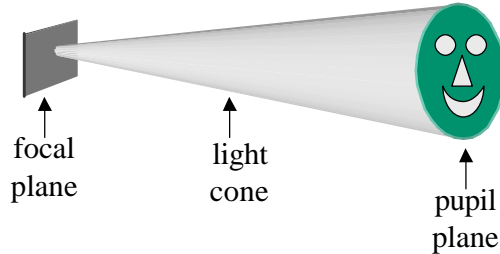
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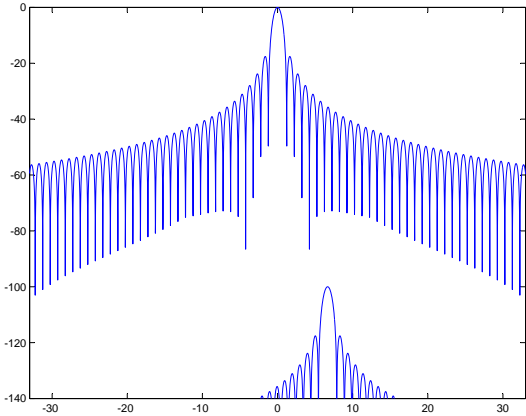
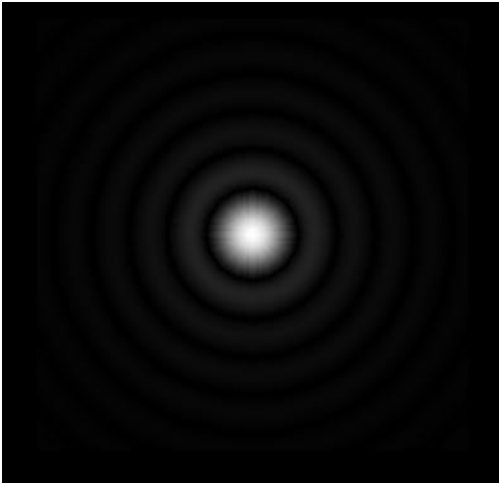
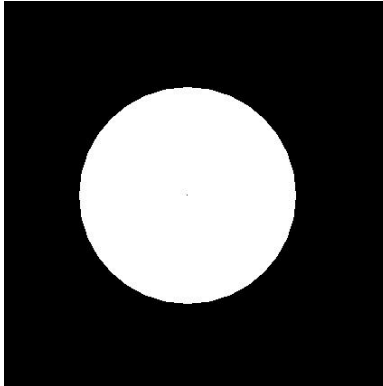
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The Airy Pattern



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Electric Field

The image-plane *electric field* $E()$ produced by an on-axis plane wave and an apodized aperture defined by an *apodization function* $A()$ is given by

$$E(\xi, \zeta) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{i(x\xi+y\zeta)} A(x, y) dy dx$$
$$\vdots$$
$$E(\rho) = 2\pi \int_0^{1/2} J_0(r\rho) A(r) r dr,$$

where J_0 denotes the 0-th order Bessel function of the first kind.

The unitless pupil-plane “length” r is given as a multiple of the aperture D .

The unitless image-plane “length” ρ is given as a multiple of focal-length times wavelength over aperture ($f\lambda/D$) or, equivalently, as an angular measure on the sky, in which case it is a multiple of just λ/D . (Example: $\lambda = 0.5\mu\text{m}$ and $D = 10\text{m}$ implies $\lambda/D = 10\text{mas}$.)

The *intensity* is the square of the electric field.



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Performance Metrics



Inner and Outer Working Angles

$$\rho_{iwa} \quad \rho_{owa}$$

Contrast:

$$E^2(\rho)/E^2(0)$$

Airy Throughput:

$$\frac{\int_0^{\rho_{iwa}} E^2(\rho) 2\pi\rho d\rho}{(\pi(1/2)^2)} = 8 \int_0^{\rho_{iwa}} E^2(\rho)\rho d\rho.$$

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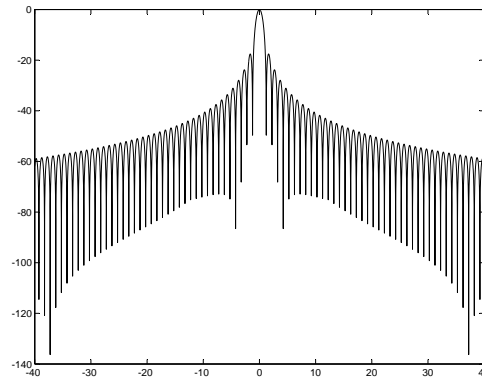
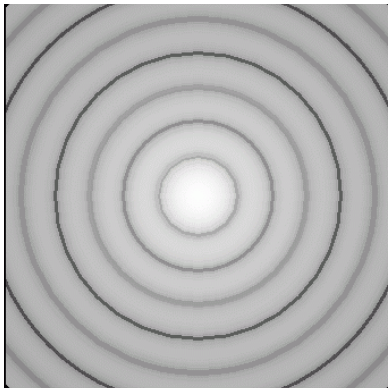
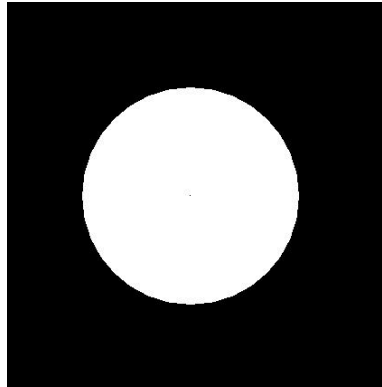
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Clear Aperture—Airy Pattern

$$\rho_{iwa} = 1.24 \quad \mathcal{T}_{\text{Airy}} = 84.2\% \quad \text{Contrast} = 10^{-2}$$

$$\rho_{iwa} = 748 \quad \mathcal{T}_{\text{Airy}} = 100\% \quad \text{Contrast} = 10^{-10}$$



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Optimization



Find *apodization* function $A()$ that solves:

$$\begin{aligned} &\text{maximize} && \int_0^{1/2} A(r) 2\pi r dr \\ &\text{subject to} && -10^{-5} E(0) \leq E(\rho) \leq 10^{-5} E(0), && \rho_{iwa} \leq \rho \leq \rho_{owa}, \\ & && 0 \leq A(r) \leq 1, && 0 \leq r \leq 1/2, \end{aligned}$$

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Optimization



Find *apodization* function $A()$ that solves:

$$\begin{aligned} &\text{maximize} && \int_0^{1/2} A(r) 2\pi r dr \\ &\text{subject to} && -10^{-5} E(0) \leq E(\rho) \leq 10^{-5} E(0), && \rho_{iwa} \leq \rho \leq \rho_{owa}, \\ & && 0 \leq A(r) \leq 1, && 0 \leq r \leq 1/2, \\ & && -50 \leq A''(r) \leq 50, && 0 \leq r \leq 1/2 \end{aligned}$$

An infinite dimensional *linear programming* problem.

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The AMPL Model



```
function J0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

param dr := (1/2)/N;
set Rs ordered := setof {j in 0.5..N-0.5 by 1} (1/2)*j/N;

var A {Rs} >= 0, <= 1, := 1/2;

set Rhos ordered := setof {j in 0..N} j*rho1/N;
set PlanetBand := setof {rho in Rhos: rho>=rho0 && rho<=rho1} rho;

var E0 {rho in Rhos} =
    2*pi*sum {r in Rs} A[r]*J0(2*pi*r*rho)*r*dr;

maximize area: sum {r in Rs} 2*pi*A[r]*r*dr;
subject to sidelobe_pos {rho in PlanetBand}: E0[rho] <= 10^(-5)*E0[0];
subject to sidelobe_neg {rho in PlanetBand}: -10^(-5)*E0[0] <= E0[rho];

subject to smooth {r in Rs: r != first(Rs) && r != last(Rs)}:
    -50*dr^2 <= A[next(r)] - 2*A[r] + A[prev(r)] <= 50*dr^2;

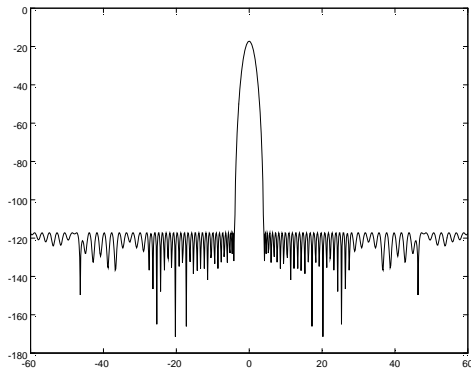
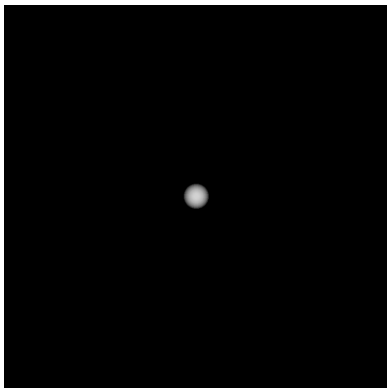
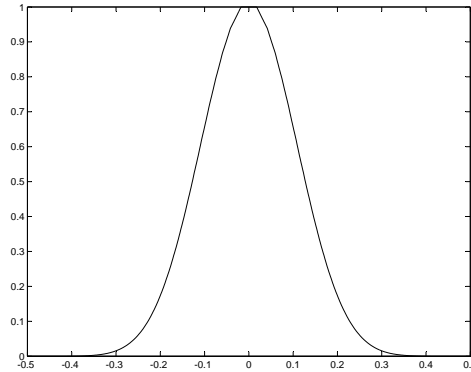
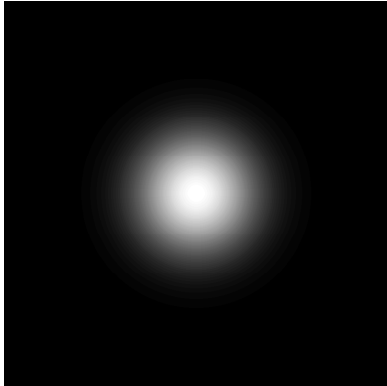
solve;
```

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The Optimal Apodization

$$\rho_{iwa} = 4 \quad \mathcal{T}_{\text{Airy}} = 9\%$$

Excellent dark zone. **Unmanufacturable.**



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Concentric Ring Masks

Recall that for circularly symmetric apodizations

$$E(\rho) = 2\pi \int_0^{1/2} J_0(r\rho)A(r)rdr,$$

where J_0 denotes the 0-th order Bessel function of the first kind.

Let

$$A(r) = \begin{cases} 1 & r_{2j} \leq r \leq r_{2j+1}, \quad j = 0, 1, \dots, m-1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$0 \leq r_0 \leq r_1 \leq \dots \leq r_{2m-1} \leq 1/2.$$

The integral can now be written as a sum of integrals and each of these integrals can be explicitly integrated to get:

$$E(\rho) = \sum_{j=0}^{m-1} \frac{1}{\rho} \left(r_{2j+1} J_1(\rho r_{2j+1}) - r_{2j} J_1(\rho r_{2j}) \right).$$



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Mask Optimization Problem

$$\text{maximize } \sum_{j=0}^{m-1} \pi(r_{2j+1}^2 - r_{2j}^2)$$

$$\text{subject to: } -10^{-5}E(0) \leq E(\rho) \leq 10^{-5}E(0), \quad \text{for } \rho_0 \leq \rho \leq \rho_1$$

where $E(\rho)$ is the function of the r_j 's given on the previous slide.

This problem is a semiinfinite nonconvex optimization problem.



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The AMPL Model



```
function intrJ0;

param pi := 4*atan(1);
param N := 400; # discretization parameter
param rho0 := 4;
param rho1 := 60;

var r {j in 0..M} >= 0, <= 1/2, := r0[j];

set Rhos2 ordered := setof {j in 0..N} (j+0.5)*rho1/N;
set PlanetBand2 := setof {rho in Rhos2: rho>=rho0 && rho<=rho1} rho;

var E {rho in Rhos2} =
    (1/(2*pi*rho)^2)*sum {j in 0..M by 2}
    (intrJ0(2*pi*rho*r[j+1]) - intrJ0(2*pi*rho*r[j]));

maximize area2: sum {j in 0..M by 2} (pi*r[j+1]^2 - pi*r[j]^2);
subject to sidelobe_pos2 {rho in PlanetBand2}: E[rho] <= 10^(-5)*E[first(rhos2)];
subject to sidelobe_neg2 {rho in PlanetBand2}: -10^(-5)*E[first(rhos2)] <= E[rho];

subject to order {j in 0..M-1}: r[j+1] >= r[j];

solve mask;
```

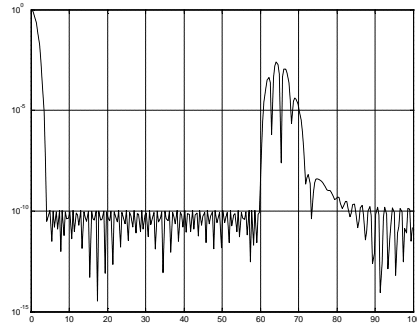
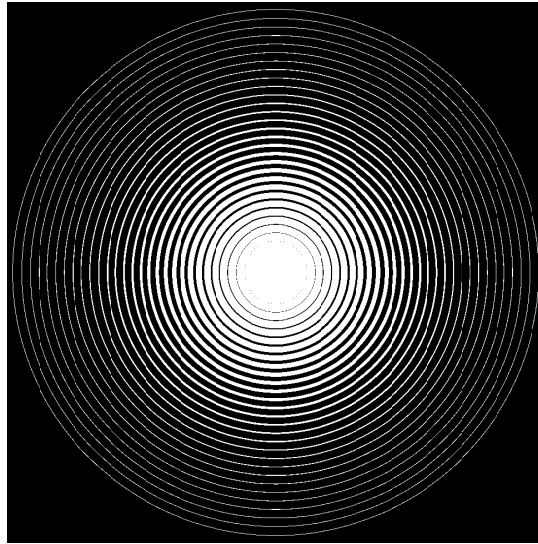
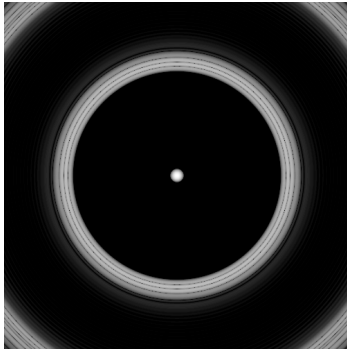
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The Best Concentric Ring Mask

$$\rho_{iwa} = 4 \quad \rho_{owa} = 60$$

$$\mathcal{T}_{\text{Airy}} = 9\%$$

Lay it on glass?



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Other Masks



Consider a binary apodization (i.e., a mask) consisting of an opening given by

$$A(x, y) = \begin{cases} 1 & |y| \leq a(x) \\ 0 & \text{else} \end{cases}$$

We only consider masks that are symmetric with respect to both the x and y axes. Hence, the function $a(\cdot)$ is a nonnegative even function.

In such a situation, the electric field $E(\xi, \zeta)$ is given by

$$\begin{aligned} E(\xi, \zeta) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-a(x)}^{a(x)} e^{i(x\xi+y\zeta)} dy dx \\ &= 4 \int_0^{\frac{1}{2}} \cos(x\xi) \frac{\sin(a(x)\zeta)}{\zeta} dx \end{aligned}$$

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Maximizing Throughput

Because of the symmetry, we only need to optimize in the first quadrant:

$$\text{maximize } 4 \int_0^{\frac{1}{2}} a(x) dx$$

$$\begin{aligned} \text{subject to } & -10^{-5} E(0,0) \leq E(\xi, \zeta) \leq 10^{-5} E(0,0), \quad \text{for } (\xi, \zeta) \in \mathcal{O} \\ & 0 \leq a(x) \leq 1/2, \quad \text{for } 0 \leq x \leq 1/2 \end{aligned}$$

The objective function is the total open area of the mask. The first constraint guarantees 10^{-10} light intensity throughout a desired region of the focal plane, and the remaining constraint ensures that the mask is really a mask.

If the set \mathcal{O} is a subset of the x -axis, then the problem is an infinite dimensional linear programming problem.



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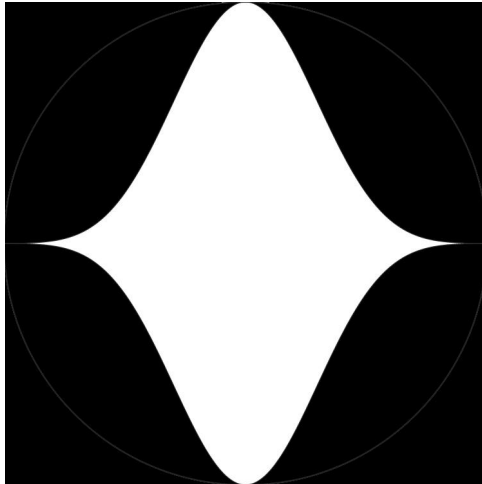
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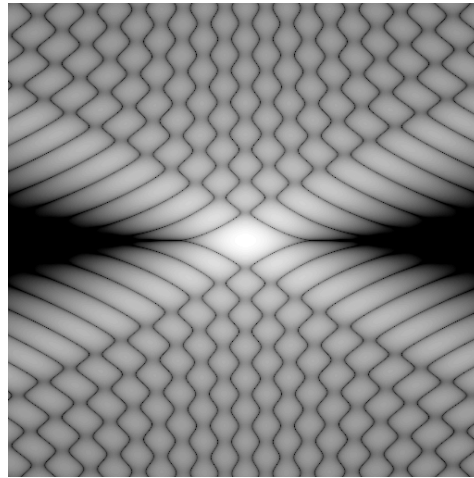
One Pupil w/ On-Axis Constraints

$$\rho_{iwa} = 4 \quad \mathcal{T}_{\text{Airy}} = 43\%$$

Small dark zone...Many rotations required



PSF for Single Prolate Spheroidal Pupil



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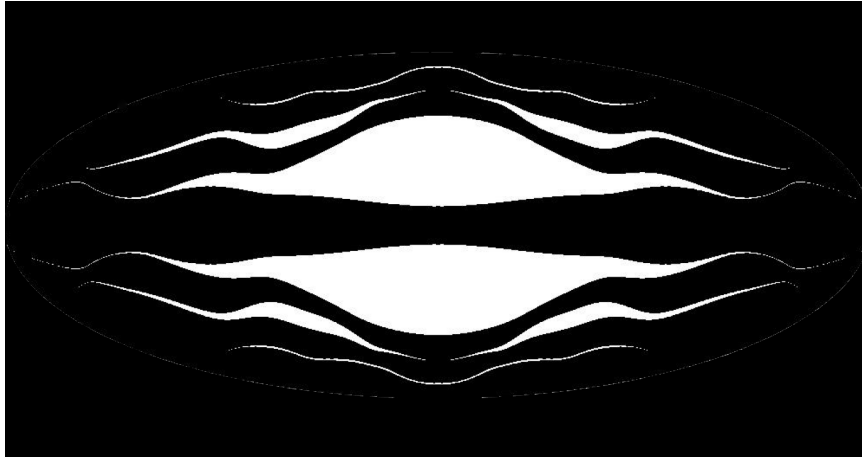
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Multiple Pupil Mask



$$\rho_{iwa} = 4$$

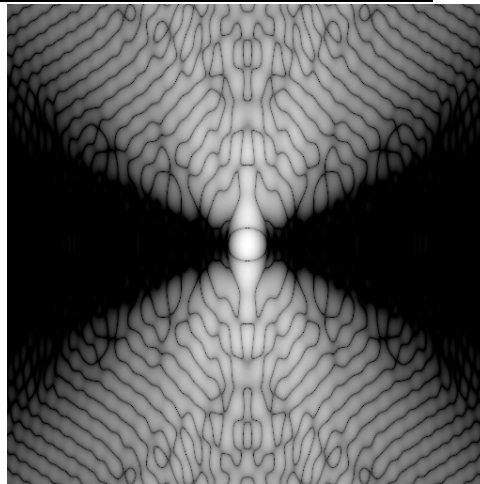
$$\mathcal{T}_{\text{Airy}} = 30\%$$

Throughput relative to ellipse

11% central obstr.

Easy to make

Only a few rotations



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Optimization Success Story

From an April 12, 2004, letter from Charles Beichman:

Dear TPF-SWG,

I am writing to inform you of exciting new developments for TPF. As part of the Presidents new vision for NASA, the agency has been directed by the President to *conduct advanced telescope searches for Earth-like planets and habitable environments around other stars*. Dan Coulter, Mike Devirian, and I have been working with NASA Headquarters (Lia LaPiana, our program executive; Zlatan Tsvetanov, our program scientist; and Anne Kinney) to incorporate TPF into the new NASA vision. The result of these deliberations has resulted in the following plan for TPF:

1. Reduce the number of architectures under study from four to two: **(a) the moderate sized coronagraph, nominally the 4x6 m version now under study**; and (b) the formation flying interferometer presently being investigated with ESA. Studies of the other two options, the large, 10-12 m, coronagraph and the structurally connected interferometer, would be documented and brought to a rapid close.
2. Pursue an approach that would result in the launch of BOTH systems within the next 10-15 years. The primary reason for carrying out two missions is the power of observations at IR and visible wavelength regions to determine the properties of detected planets and to make a reliable and robust determination of habitability and the presence of life.
3. **Carry out a modest-sized coronagraphic mission, TPF-C, to be launched around 2014**, to be followed by a formation-flying interferometer, TPF-I, to be conducted jointly with ESA and launched by the end of decade (2020). This ordering of missions is, of course, subject to the readiness of critical technologies and availability of funding. But in the estimation of NASA HQ and the project, the science, the technology, the political will, and the budgetary resources are in place to support this plan.

...

The opportunity to move TPF forward as part of the new NASA vision has called for these rapid and dramatic actions. What has made these steps possible has been the hard work by the entire team, including the TPF-SWG, the two *TPF architecture teams*, and all the technologists at JPL and around the country, which has demonstrated that NASA is ready to proceed with both TPF-C and TPF-I and that the data from these two missions are critical to the success of the goals of TPF. We will be making more information available as soon as additional details become available. Thank you for all your help in preparing TPF to take advantage of this opportunity.



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The N -Body Problem



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Least Action Principle



Given: n bodies.

Let:

m_j denote the mass and

$z_j(t)$ denote the position in $\mathbb{R}^2 = \mathbb{C}$ of body j at time t .

Action Functional:

$$A = \int_0^{2\pi} \left(\sum_j \frac{m_j}{2} \|\dot{z}_j\|^2 + \sum_{j,k:k < j} \frac{m_j m_k}{\|z_j - z_k\|} \right) dt.$$

Minimize!

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Equations of Motion



First Variation:

$$\begin{aligned}\delta A &= \int_0^{2\pi} \sum_{\alpha} \left(\sum_j m_j \dot{z}_j^{\alpha} \delta z_j^{\alpha} - \sum_{j,k:k < j} m_j m_k \frac{(z_j^{\alpha} - z_k^{\alpha})(\delta z_j^{\alpha} - \delta z_k^{\alpha})}{\|z_j - z_k\|^3} \right) dt \\ &= - \int_0^{2\pi} \sum_j \sum_{\alpha} \left(m_j \ddot{z}_j^{\alpha} + \sum_{k:k \neq j} m_j m_k \frac{z_j^{\alpha} - z_k^{\alpha}}{\|z_j - z_k\|^3} \right) \delta z_j^{\alpha} dt\end{aligned}$$

Setting first variation to zero, we get:

$$m_j \ddot{z}_j^{\alpha} = - \sum_{k:k \neq j} m_j m_k \frac{z_j^{\alpha} - z_k^{\alpha}}{\|z_j - z_k\|^3}, \quad j = 1, 2, \dots, n, \quad \alpha = 1, 2$$

Note: If $m_j = 0$ for some j , then the first order optimality condition reduces to $0 = 0$, which is *not* the equation of motion for a massless body.

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Periodic Solutions

We assume solutions can be expressed in the form

$$z_j(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ikt}, \quad \gamma_k \in \mathbb{C}.$$

Writing with components $z_j(t) = (x_j(t), y_j(t))$ and $\gamma_k = (\alpha_k, \beta_k)$, we get

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k^c \cos(kt) + a_k^s \sin(kt))$$

$$y(t) = b_0 + \sum_{k=1}^{\infty} (b_k^c \cos(kt) + b_k^s \sin(kt))$$

where

$$\begin{aligned} a_0 &= \alpha_0, & a_k^c &= \alpha_k + \alpha_{-k}, & a_k^s &= \beta_{-k} - \beta_k, \\ b_0 &= \beta_0, & b_k^c &= \beta_k + \beta_{-k}, & b_k^s &= \alpha_k - \alpha_{-k}. \end{aligned}$$

The variables a_0 , a_k^c , a_k^s , b_0 , b_k^c , and b_k^s are the decision variables in the optimization model.



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The AMPL Model. Too hard!!



```
param N := 3; # number of masses
param n := 15; # number of terms in Fourier series representation
param m := 100; # number of terms in numerical approx to integral

param theta {j in 0..m-1} := j*2*pi/m;

var x {i in 0..N-1, j in 0..m-1};
var y {i in 0..N-1, j in 0..m-1};

var xdot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (x[i,j+1]-x[i,j])*m/(2*pi) else (x[i,0]-x[i,m-1])*m/(2*pi);
var ydot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (y[i,j+1]-y[i,j])*m/(2*pi) else (y[i,0]-y[i,m-1])*m/(2*pi);

var K {j in 0..m-1} = 0.5*sum {i in 0..N-1} (xdot[i,j]^2 + ydot[i,j]^2);

var P {j in 0..m-1}
  = - sum {i in 0..N-1, ii in 0..N-1: ii>i}
    1/sqrt((x[i,j]-x[ii,j])^2 + (y[i,j]-y[ii,j])^2);

minimize A: (2*pi/m)*sum {j in 0..m-1} (K[j] - P[j]);
```

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The AMPL Model. Tractable



```
param N := 3; # number of masses
param n := 15; # number of terms in Fourier series representation
param m := 100; # number of terms in numerical approx to integral

param theta {j in 0..m-1} := j*2*pi/m;

param a0 {i in 0..N-1} default 0;          param b0 {i in 0..N-1} default 0;
var as {i in 0..N-1, k in 1..n} := 0;      var bs {i in 0..N-1, k in 1..n} := 0;
var ac {i in 0..N-1, k in 1..n} := 0;      var bc {i in 0..N-1, k in 1..n} := 0;

var x {i in 0..N-1, j in 0..m-1}
  = a0[i]+sum {k in 1..n} ( as[i,k]*sin(k*theta[j]) + ac[i,k]*cos(k*theta[j]) );
var y {i in 0..N-1, j in 0..m-1}
  = b0[i]+sum {k in 1..n} ( bs[i,k]*sin(k*theta[j]) + bc[i,k]*cos(k*theta[j]) );

var xdot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (x[i,j+1]-x[i,j])*m/(2*pi) else (x[i,0]-x[i,m-1])*m/(2*pi);
var ydot {i in 0..N-1, j in 0..m-1}
  = if (j<m-1) then (y[i,j+1]-y[i,j])*m/(2*pi) else (y[i,0]-y[i,m-1])*m/(2*pi);

var K {j in 0..m-1} = 0.5*sum {i in 0..N-1} (xdot[i,j]^2 + ydot[i,j]^2);

var P {j in 0..m-1}
  = - sum {i in 0..N-1, ii in 0..N-1: ii>i}
    1/sqrt((x[i,j]-x[ii,j])^2 + (y[i,j]-y[ii,j])^2);

minimize A: (2*pi/m)*sum {j in 0..m-1} (K[j] - P[j]);
```

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Continued...

```
let {i in 0..N-1, k in 1..n} as[i,k] := 1*(Uniform01()-0.5);
let {i in 0..N-1, k in 1..n} ac[i,k] := 1*(Uniform01()-0.5);
let {i in 0..N-1, k in n..n} bs[i,k] := 0.01*(Uniform01()-0.5);
let {i in 0..N-1, k in n..n} bc[i,k] := 0.01*(Uniform01()-0.5);

solve;
```



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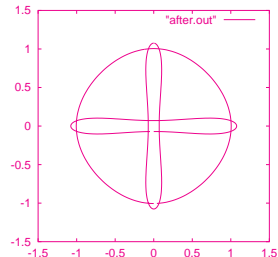
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Choreographies and the Ducati



The previous AMPL model was used to find many *choreographies* (a la **Moore** and Montgomery/Chencinier) in the equimass n -body problem and the stable *Ducati* solution to the 3-body problem.



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Antenna Array Design



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Antenna Arrays

- Given: an array (linear or 2-D) of radar antennae.
- An incoming signal induces an output signal at each antenna.
- A linear combination of the signals is made to produce one total output signal.
- Coefficients of the linear combination can be chosen to accentuate and/or attenuate the output signal's strength as a function of the input signal's source direction.



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2-D Antenna-Array Design Problem

$$\begin{aligned} & \text{minimize} && \int_S |A(p)|^2 ds \\ & \text{subject to} && A(p_0) = 1, \end{aligned}$$

where

$$A(p) = \sum_{l \in \{\text{array elements}\}} w_l e^{-2\pi i p \cdot x_l}, \quad p \in S$$

w_l = complex-valued **design weight** for array element l

S = subset of unit hemisphere: sidelobe directions

x_l = spatial coord vector for array element l

p_0 = “look” direction



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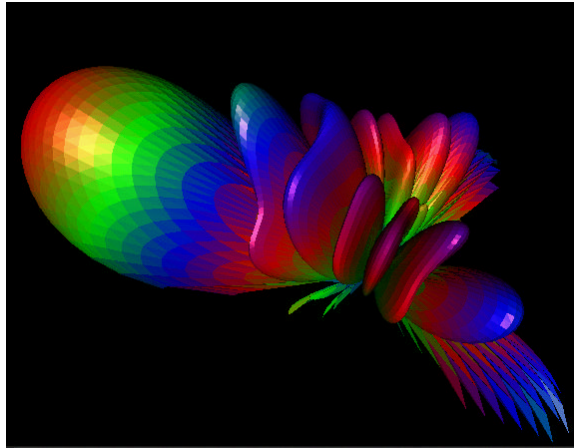
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Specific Example: Hexagonal Lattice of 61 Elements

$$\rho = -20 \text{ dB} = 0.01$$

$S = 889$ points outside 20° from look direction

$p_0 = 40^\circ$ from zenith



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Audio Filter Design



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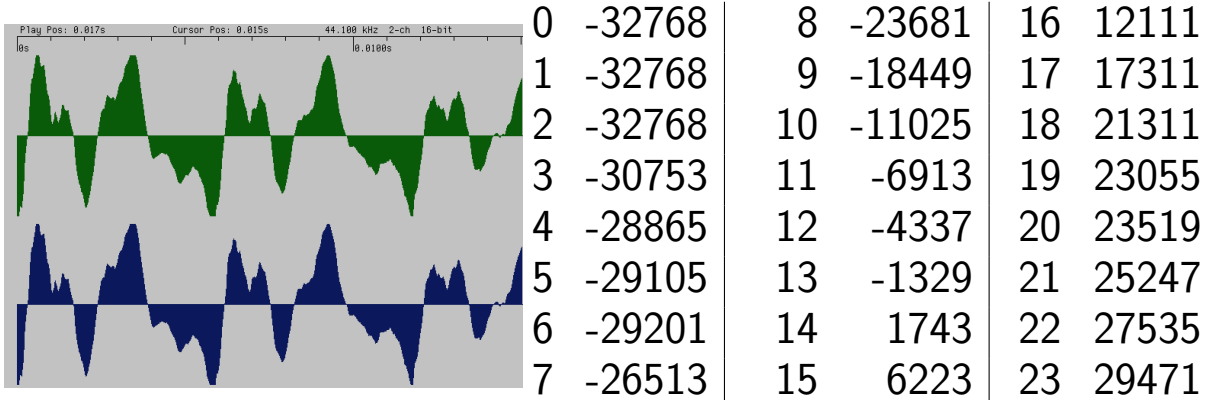
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Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: $u_k, k \in \mathbb{Z}$.
- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.
- For CD quality sound, 44100 short integers get played per second per channel.



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FIR Filter Design—Continued

- A *finite impulse response (FIR) filter* takes as input a digital signal and convolves this signal with a finite set of fixed numbers h_0, \dots, h_n to produce a filtered output signal:

$$y_k = \sum_{i=-n}^n h_{|i|} u_{k-i}.$$

- Sparring the details, the output power at frequency ν is given by

$$|H(\nu)|^2$$

where

$$H(\nu) = \sum_{k=-n}^n h_{|k|} e^{2\pi i k \nu} = h(0) + 2 \sum_{k=1}^n h_k \cos(2\pi k \nu),$$

- Similarly, the mean absolute deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0, 1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} |H(\nu) - 1| d\nu$$



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Filter Design: Woofer, Midrange, Tweeter

$$\begin{aligned} \text{minimize} \quad & \int_0^1 |H_w(\nu) + H_m(\nu) + H_t(\nu) - 1| d\nu \\ \text{subject to} \quad & -\epsilon \leq H_w(\nu) \leq \epsilon, \quad \nu \in W = [.2, .8] \\ & -\epsilon \leq H_m(\nu) \leq \epsilon, \quad \nu \in M = [.4, .6] \cup [.9, .1] \\ & -\epsilon \leq H_t(\nu) \leq \epsilon, \quad \nu \in T = [.7, .3] \end{aligned}$$

where

$$H_i(\nu) = h_0^i + 2 \sum_{k=1}^n h_k^i \cos(2\pi k\nu), \quad i = W, M, T$$

h_k^i = filter coefficients, i.e., **decision variables**



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Conversion to a Linear Programming Problem

$$\text{minimize } \int_0^1 t(\nu) d\nu$$

$$\text{subject to } t(\nu) \leq H_w(\nu) + H_m(\nu) + H_t(\nu) - 1 \leq t(\nu) \quad \nu \in [0, 1]$$

$$-\epsilon \leq H_w(\nu) \leq \epsilon, \quad \nu \in W$$

$$-\epsilon \leq H_m(\nu) \leq \epsilon, \quad \nu \in M$$

$$-\epsilon \leq H_t(\nu) \leq \epsilon, \quad \nu \in T$$



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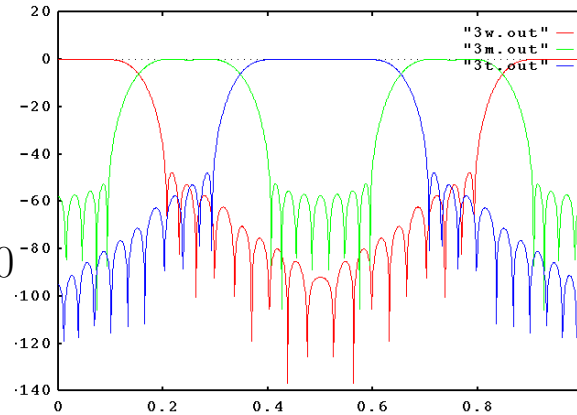
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Specific Example

filter length: $n = 14$

integral discretization: $N = 1000$



Demo: [orig-clip](#) [woofer](#) [midrange](#) [tweeter](#) [reassembled](#)

Ref: J.O. Coleman, U.S. Naval Research Laboratory,

CISS98 paper available: enr.umbc.edu/~jeffc/pubs/abstracts/ciss98.html



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