THE 2-BODY PROBLEM

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ABSTRACT. In this short note, we show that a pair of ellipses with a common focus is a solution to the 2-body problem

1. INTRODUCTION.

Solving the 2-body problem from scratch is doable but difficult. But, what if we simply want to verify that there are elliptical orbits where the two ellipses share a common focus, which is also the center of mass of the system? With these suppositions, maybe this problem isn't so bad. In other words, let's try to use *guess-n-check* method. It should be easier. (We all believe that P $\neq$ NP, right!) Here we go...

First, choose the coordinate system so that the foci lie on the horizontal x-axis and so that the shared focus is at the origin (see Figure).

One body's second focus is on the positive x-axis and the other body's second focus is on the negative x-axis. Call the right body "body one" and the left body will be "body two". The orbit of

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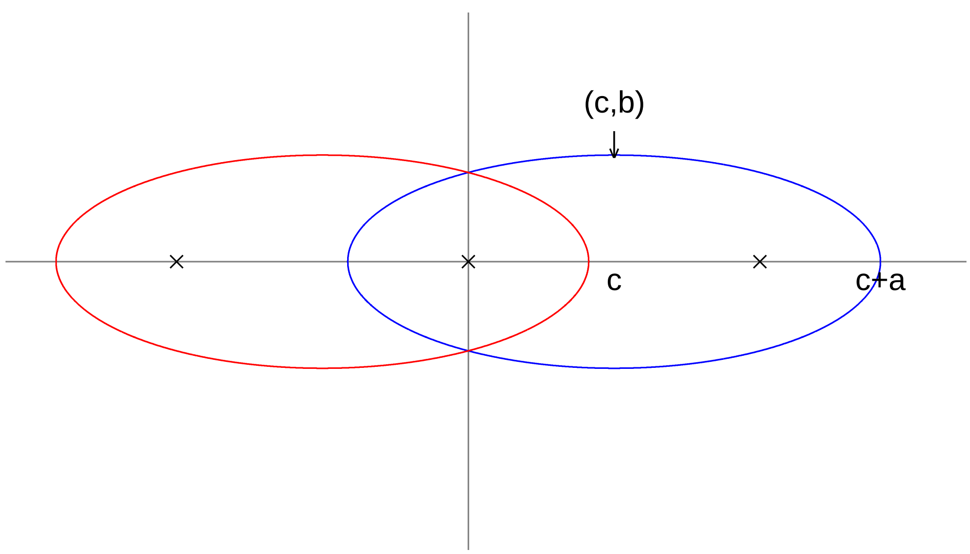

FIGURE 1. A pair of ellipses sharing a common focus.

body one can be given parametrically as

$$\begin{aligned} x_1 &= c + a\cos\theta \\ y_1 &= b\sin\theta. \end{aligned}$$

Here, a, b, and c are constants whereas x_1, y_1, and θ are functions of time t. The constant c is the x-coordinate of the center of the ellipse and the constants a and b are the semi-major and semi-minor axes, respectively. Clearly all three constants are positive numbers and $a > b$. Furthermore, an important property of ellipses is that the distance from the center of the ellipse to a focus is $\sqrt{a^2 - b^2}$. Since $(c, 0)$ is the center of the ellipse and $(0, 0)$ is a focus, it follows that

$$c = \sqrt{a^2 - b^2}.$$

The assumption that the center of mass of the system coincides with the focus at the origin implies that

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

The distance r between the two bodies plays and important role in Newton's law of gravity, so we start by computing it:

$$\begin{aligned} r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2x_1)^2 + (2y_1)^2} \\ &= 2\sqrt{x_1^2 + y_1^2} \\ &= 2\sqrt{(c + a\cos\theta)^2 + (b\sin\theta)^2} \\ &= 2\sqrt{c^2 + 2ac\ \cos\theta + a^2\cos^2\theta + b^2\sin^2\theta} \\ &= 2\sqrt{c^2 + 2ac\ \cos\theta + a^2\cos^2\theta + (a^2 - c^2)\sin^2\theta} \\ &= 2\sqrt{c^2\cos^2\theta + 2ac\ \cos\theta + a^2} \\ &= 2(a + c\ \cos\theta). \end{aligned}$$

Newton's laws involve accelerations and so we differentiate once

$$\begin{aligned} \dot{x}_1 &= -a\sin\theta\,\dot{\theta} \\ \dot{y}_1 &= \quad b\cos\theta\,\dot{\theta}. \end{aligned}$$

and then a second time

$$\begin{aligned} \ddot{x}_1 &= -a\sin\theta\,\ddot{\theta} - a\cos\theta\,\dot{\theta}^2 \\ \ddot{y}_1 &= \quad b\cos\theta\,\ddot{\theta} - b\sin\theta\,\dot{\theta}^2. \end{aligned}$$

For simplicity, and without loss of generality, suppose that units are chosen in such a way that the gravitational constant G equals one. Also for simplicity, but with some loss of generality, suppose that the masses of the two bodies are both equal to one. With these assumptions, Newton's law of gravity is

$$\ddot{x}_1 = \frac{x_2 - x_1}{r^3} = \frac{-x_1}{4(a + c\ \cos\theta)^3}$$

$$\ddot{y}_1 = \frac{y_2 - y_1}{r^3} = \frac{-y_1}{4(a + c\ \cos\theta)^3}.$$

Substituting the formulas for x_1, y_1, and their second derivatives derived above, we get

$$-a\sin\theta\,\ddot{\theta} - a\cos\theta\,\dot{\theta}^2 = \frac{-(c + a\cos\theta)}{4(a + c\ \cos\theta)^3}$$

$$b\cos\theta\,\ddot{\theta} - b\sin\theta\,\dot{\theta}^2 = \frac{-(b\sin\theta)}{4(a + c\ \cos\theta)^3}.$$

Negating both sides and writing in matrix form, we see that

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix} = \begin{bmatrix} \dfrac{c/a + \cos\theta}{4(a + c\ \cos\theta)^3} \\ \dfrac{\sin\theta}{4(a + c\ \cos\theta)^3} \end{bmatrix}$$

Multiplying by the matrix inverse, we get

$$\begin{aligned} \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix} &= \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \dfrac{c/a + \cos\theta}{4(a + c\ \cos\theta)^3} \\ \dfrac{\sin\theta}{4(a + c\ \cos\theta)^3} \end{bmatrix} \\ &= \frac{1}{4(a + c\ \cos\theta)^3} \begin{bmatrix} \sin\theta(c/a + \cos\theta) - \cos\theta\sin\theta \\ \cos\theta(c/a + \cos\theta) + \sin\theta\sin\theta \end{bmatrix} \\ &= \frac{1}{4(a + c\ \cos\theta)^3} \begin{bmatrix} (c/a)\sin\theta \\ (c/a)\cos\theta + 1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{c\sin\theta}{4a(a + c\ \cos\theta)^3} \\ \dfrac{1}{4a(a + c\ \cos\theta)^2} \end{bmatrix} \end{aligned}$$

In other words, for a solution to exist, the function θ must simultaneously be a solution to both of the following, possibly inconsistent, equations:

$$\ddot{\theta} = \frac{c\sin\theta}{4a(a + c\ \cos\theta)^3}$$

$$\dot{\theta} = \frac{1}{2\sqrt{a}(a + c\ \cos\theta)}$$

To verify consistency, we check that differentiating the equation for $\dot\theta$ brings us to the equation derived above for $\ddot\theta$:

$$\begin{aligned}\ddot\theta &= \frac{1}{2\sqrt{a}}\ \frac{c\ \sin\theta\ \dot\theta}{(a+c\ \cos\theta)^2} \\ &= \frac{1}{2\sqrt{a}}\ \frac{c\ \sin\theta}{(a+c\ \cos\theta)^2}\ \frac{1}{2\sqrt{a}(a+c\ \cos\theta)} \\ &= \frac{c\sin\theta}{4a(a+c\ \cos\theta)^3}.\end{aligned}$$

Hence, we see that the formulas for the first and second derivative of θ are indeed consistent with each other.
Picking $\theta(0)$ can be given any value as it merely determines where the bodies are in their orbit at time $t = 0$. The semi-major and semi-minor axes a and b determine c (as already mentioned) and $\dot\theta(0)$:

$$\dot\theta(0) = \frac{c\sin\theta(0)}{4a(a+c\ \cos\theta(0))^3}.$$

Suppose, for simplicity, that $\theta(0) = 0$. The differential equation for $\dot\theta(t)$ can be rewritten as

$$(a + c\cos\theta)\,d\theta = \frac{dt}{2\sqrt{a}}.$$

Since $\varepsilon = c/a$ is the well-known *eccentricity* of the ellipse, it is common to divide both sides of this equation by a thereby replacing c with the eccentricity ε:

$$(1 + \varepsilon\cos\theta)\,d\theta = \frac{dt}{2a^{3/2}}.$$

Integrating from 0 to t, we get

$$\int_0^{\theta(t)} (1 + \varepsilon\cos\theta)\,d\theta = \int_0^t \frac{dt}{2a^{3/2}}.$$

The integrals can be computed explicitly. The result is

$$\theta(t) + \varepsilon\sin\theta(t) = \frac{t}{2a^{3/2}}.$$

Unfortunately, this is a transcendental equation for $\theta(t)$ and so it does not have a simple "closed form" solution. However, in the case where the eccentricity is zero, we have an exact solution:

$$\theta(t) = \frac{t}{2a^{3/2}}.$$

And, when the eccentricity is not zero, the following recursion quickly converges on the correct answer:

$$\begin{aligned}
\theta^{(0)}(t) &= 0 \\
\theta^{(1)}(t) &= \frac{t}{2a^{3/2}} - \varepsilon \sin \theta^{(0)}(t) \\
&\vdots \\
\theta^{(k+1)}(t) &= \frac{t}{2a^{3/2}} - \varepsilon \sin \theta^{(k)}(t) \\
&\vdots
\end{aligned}$$

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