

Frontiers of Stochastically Nondominated Portfolios

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ABSTRACT

- A portfolio is *bad* if another portfolio dominates it (stochastically).
- Many portfolios on Markowitz's "efficient frontier" are bad.
- We give alternative risk measures—the associated efficient frontiers are good.
- Furthermore, these new models can be formulated as LPs (instead of Markowitz's QPs).
- Even more, using the parametric simplex method the entire efficient frontier can be computed in the time normally required to find just one point on the frontier.
- Lastly, our efficient frontier is completely determined by a finite set of portfolios (vs. a continuum).



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Portfolio Optimization

Markowitz Shares the 1990 Nobel Prize



Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences
in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS
AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize
in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called,
Capital Asset Pricing Model (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary

Financial markets serve a key purpose in a modern market economy by allocating productive resources
among various areas of production. It is to a large extent through financial markets that saving in
different sectors of the economy is transferred to firms for investments in buildings and machines.
Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread
and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry
Markowitz who developed a theory for households' and firms' allocation of financial assets under
uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally
invested in assets which differ in regard to their expected return and risk, and thereby also how risks can
be reduced.



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Why Make a Portfolio? ... Hedging



Investment A: up 20%, down 10%, equally likely—a risky asset.

Investment B: up 20%, down 10%, equally likely—another risky asset.

Correlation: up years for A are down years for B and vice versa.

Portfolio—half in A, half in B: up 5% every year! No risk!

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Historical Data



Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

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Notation: $R_j(t)$ = return on investment j in time period t .

The Ingredients: Risk and Reward



Raw Data:

$R_j(t)$ = return on asset j
in time period t



Derived Data:

$$\mu_j = \frac{1}{T} \sum_{t=1}^T R_j(t)$$
$$D_{tj} = R_j(t) - \mu_j.$$

Decision Variables:

x_j = fraction of portfolio
to invest in asset j

$$R(x) = \sum_j x_j R_j$$

Decision Criteria:

$$\mu(x) = \sum_j \mu_j x_j$$

$$\rho_2(x) = \frac{1}{T} \sum_{t=1}^T \left(\sum_j D_{tj} x_j \right)^2$$

$$\rho_1(x) = \frac{1}{T} \sum_{t=1}^T \left| \sum_j D_{tj} x_j \right|$$

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Portfolio Optimization Problems



Set a value for *risk aversion* parameter λ and maximize a combination of reward and negative-risk.

Markowitz (quadratic!) model:

$$\text{maximize } \lambda\mu(x) - \rho_2(x)$$

$$\text{subject to } \sum_j x_j = 1 \quad \text{and} \quad x_j \geq 0 \text{ for all investments } j$$

Replacing variance ρ_2 with *mean absolute deviation (MAD)*, ρ_1 , yields a linearizeable model:

$$\text{maximize } \lambda\mu(x) - \rho_1(x)$$

$$\text{subject to } \sum_j x_j = 1 \quad \text{and} \quad x_j \geq 0 \text{ for all investments } j$$

The MAD formulation has both theoretical and practical advantages.

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Stochastic Dominance

Given: two random variables, V and S .

- **First order stochastic dominance:** $V \succeq_1 S$ means

$$F_V(\eta) \leq F_S(\eta), \quad \text{for all } \eta \in \mathbb{R},$$

where F_V and F_S denote the cumulative distribution functions of V and S , respectively:

$$F_V(\eta) = \mathbb{P}\{V \leq \eta\} \quad \text{for } \eta \in \mathbb{R}.$$

- **Second order stochastic dominance:** $V \succeq_2 S$ means

$$\int_{-\infty}^{\eta} F_V(\xi) d\xi \leq \int_{-\infty}^{\eta} F_S(\xi) d\xi, \quad \text{for all } \eta \in \mathbb{R}.$$



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Stochastic Dominance and Utility Theory

Second order stochastic dominance characterizes those random variables that every risk averse decision maker would prefer to a given random variable:

Theorem (Levy '92) $V \succeq_2 S$ if and only if $\mathbb{E}(U(V)) \geq \mathbb{E}(U(S))$ for every nondecreasing concave function $U(\cdot)$.

Stochastic Dominance and Variance (Markowitz)

Theorem (Trivial '02) There exists $\lambda_0 \leq \infty$ such that $R(x) \succeq_2 R(y)$ implies $\lambda\mu(x) - \rho_2(x) \geq \lambda\mu(y) - \rho_2(y)$ for all $\lambda \geq \lambda_0$.

Note: λ_0 is data dependent—there is no simple a-priori estimate.

Stochastic Dominance and Mean Absolute Deviation

Theorem (Ogryczak and Ruszczyński '99) $R(x) \succeq_2 R(y)$ implies $\lambda\mu(x) - \rho_1(x) \geq \lambda\mu(y) - \rho_1(y)$ for all $\lambda \geq 2$.



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Mean Absolute Deviation Problem

$$\begin{aligned} &\text{maximize} && \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^T \left| \sum_j D_{tj} x_j \right| \\ &\text{subject to} && \sum_j x_j = 1 \\ &&& x_j \geq 0 \quad \text{for all investments } j \end{aligned}$$

Not a linear programming problem. But it's easy to convert...



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A Linear Programming Formulation



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$$\text{maximize} \quad \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^T y_t$$

$$\text{subject to} \quad -y_t \leq \sum_j D_{tj} x_j \leq y_t \quad \text{for all times } t$$

$$\sum_j x_j = 1$$

$$x_j \geq 0$$

$$y_t \geq 0$$

for all investments j

for all times t

Adding Slack Variables w_t^+ and w_t^-

$$\text{maximize } \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^T y_t$$

$$\text{subject to } -y_t - \sum_j D_{tj} x_j + w_t^- = 0 \quad \text{for all times } t$$

$$-y_t + \sum_j D_{tj} x_j + w_t^+ = 0 \quad \text{for all times } t$$

$$\sum_j x_j = 1$$

$$x_j \geq 0 \quad \text{for all investments } j$$

$$y_t, w_t^-, w_t^+ \geq 0 \quad \text{for all times } t$$



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The Solution for Large λ



Varying the risk bound $0 \leq \lambda < \infty$ produces the *efficient frontier*.

Large values of λ favor reward whereas small values favor minimizing risk.

Beyond some finite threshold value for λ , the optimal solution will be a portfolio consisting of just one asset—the asset j^* with the largest average return:

$$\mu_{j^*} \geq \mu_j \quad \text{for all } j.$$

It's easy to identify basic vs. nonbasic variables:

- Variable x_{j^*} is basic whereas the remaining x_j 's are nonbasic.
- All of the y_t 's are basic.
- If $D_{tj^*} > 0$, then w_t^- is basic and w_t^+ is nonbasic. Otherwise, it is switched.

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The Basic Optimal Solution for Large λ



Let

$$T^+ = \{t : D_{tj^*} > 0\}, \quad T^- = \{t : D_{tj^*} < 0\}, \quad \text{and} \quad \epsilon_t = \begin{cases} 1, & \text{for } t \in T^+ \\ -1, & \text{for } t \in T^- \end{cases}$$

It's tedious, but here's the optimal dictionary:

$$\begin{aligned} \zeta &= \frac{1}{T} \sum_{t=1}^T \epsilon_t D_{tj^*} + \lambda \mu_{j^*} - \frac{1}{T} \sum_{j \neq j^*} \sum_{t=1}^T \epsilon_t (D_{tj} - D_{tj^*}) x_j + \lambda \sum_{j \neq j^*} (\mu_j - \mu_{j^*}) x_j - \frac{1}{T} \sum_{t \in T^-} w_t^- - \frac{1}{T} \sum_{t \in T^+} w_t^+ \\ \hline y_t &= -D_{tj^*} - \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^- & t \in T^- \\ w_t^- &= 2D_{tj^*} + 2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^+ & t \in T^+ \\ y_t &= D_{tj^*} + \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^+ & t \in T^+ \\ w_t^+ &= -2D_{tj^*} - 2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^- & t \in T^- \\ x_{j^*} &= 1 - \sum_{j \neq j^*} x_j \end{aligned}$$

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Efficient Frontier



Varying risk bound λ produces the so-called *efficient frontier*.

λ	US 3-Month T-Bills	S&P 500	Lehman Bros. Corp. Bonds	NASDAQ Comp.	Wilshire 5000	Gold	EAFE	Reward (pct)	Risk
17.872							100.0	14.1	0.184
6.236						8.0	92.0	14.0	0.166
2.555					25.6	29.2	45.1	13.3	0.115
1.863			1.2		51.3	35.4	12.1	10.9	0.058
1.788		2.8	58.7			18.4	20.1	10.9	0.057
1.778			61.4			17.4	21.2	10.3	0.047
1.609	22.3		45.6			13.5	18.6	9.3	0.029
1.417	51.5		29.6		2.1	7.2	9.6	9.2	0.029
0.180	55.0		26.8	1.3		6.5	10.4	8.0	0.022
0.000	92.6		4.7			0.9	1.8	7.9	0.021

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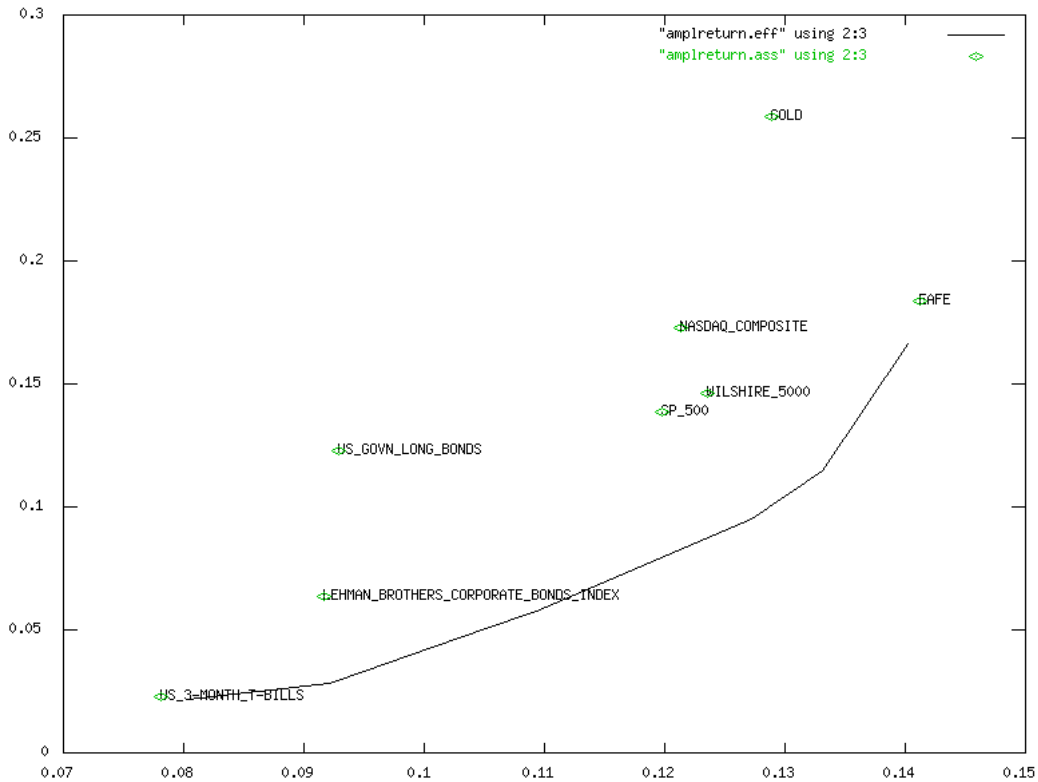
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Computing the Efficient Frontier



Using a reasonably efficient code for the *parametric self-dual simplex method* (simpo), it took *22,000* pivots and *1.5 hours* to solve for *one point* on the efficient frontier.

Customizing this same code to solve parametrically for every point on the efficient frontier, it took *23,446* pivots and *57 minutes* to compute *every point* on the frontier.

The efficient frontier consists of *23,446* distinct portfolios. Click [here](#) for a partial list (*warning: the file is 2.5 MBytes*). The complete list makes a 37 MByte file.

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Upside Risk—An Oxymoron



$$\rho_2(x) = \frac{1}{T} \sum_{t=1}^T \left(\sum_j D_{tj} x_j \right)^2 \Rightarrow \rho_2^-(x) = \frac{1}{T} \sum_{t=1}^T \left(\sum_j D_{tj} x_j \right)_-^2$$

$$\rho_1(x) = \frac{1}{T} \sum_{t=1}^T \left| \sum_j D_{tj} x_j \right| \Rightarrow \rho_1^-(x) = \frac{1}{T} \sum_{t=1}^T \left(\sum_j D_{tj} x_j \right)_-$$

where $(x)_- = \min(x, 0)$.

Theorem (*Trivial '02*) $\rho_1^-(x) = \rho_1(x)/2$.

Corollary *Efficient frontier is “good” for all $\lambda \geq 1$.*

Note: No analogous result for semi-variance.

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Means, Medians, and Quantiles



Mean is solution to: $\min_z \sum_j (b_j - z)^2$

Median is solution to: $\min_z \sum_j |b_j - z|$

In MAD portfolio model, replace

$$\left| \sum_j R_j(t)x_j - \mu(x) \right| \text{ with } \left| \sum_j R_j(t)x_j - z \right|$$

to get mean absolute deviation from the median.

Theorem *Efficient frontier is “good” for all $\lambda \geq 1$.*

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REVIEW

- A portfolio is *bad* if another portfolio dominates it (stochastically).
- Many portfolios on Markowitz's "efficient frontier" are bad.
- We gave alternative risk measures whose associated efficient frontiers are good.
- Furthermore, these new models can be formulated as LPs (instead of Markowitz's QPs).
- Even more, using the parametric simplex method the entire efficient frontier can be computed in the time normally required to find just one point on the frontier.
- Lastly, our efficient frontier is completely determined by a finite set of portfolios (vs. a continuum).

Full text of the tech report is available at:
<http://www.princeton.edu/~rvdb/tex/lpport/lpport8.pdf>



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