Frontiers of Stochastically Nondominated Portfolios

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ABSTRACT

1. The traditional (quadratic) Markowitz model produces portfolios that are stochastically dominated by portfolios not on the efficient frontier.
2. Replacing the quadratic risk measure with a mean absolute deviation (MAD) measure corrects this defect.
3. The MAD model can be formulated as a parametric linear programming problem (the risk parameter λ is the parameter).
4. The *parametric* simplex method (described in detail in my book) can be used with λ as the parameter of the parametric method.
5. Doing so, one finds ALL portfolios on the efficient frontier in roughly the same time as it takes to find just one portfolio (corresponding, say, to $\lambda = 0$).
6. The speedup is huge.

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Historical Data

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Notation: $R_j(t)$ = return on investment j in time period t.

The Ingredients: Risk and Reward

Raw Data: $\Longrightarrow$ Derived Data:

$$R_j(t) \;=\; \text{return on asset } j \text{ in time period } t$$

$$\mu_j = \frac{1}{T}\sum_{t=1}^{T} R_j(t)$$

$$D_{tj} = R_j(t) - \mu_j.$$

Decision Variables:

$$x_j \;=\; \text{fraction of portfolio to invest in asset } j$$

$$R(x) = \sum_j x_j R_j$$

Decision Criteria:

$$\mu(x) = \sum_j \mu_j x_j$$

$$\rho_2(x) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j D_{tj}x_j\right)^2$$

$$\rho_1(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j D_{tj}x_j\right|$$

Quadratic Markowitz Problem

$$\begin{aligned} \text{maximize} \quad & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left(\sum_j D_{tj} x_j \right)^2 \\ \text{subject to} \quad & \sum_j x_j = 1 \\ & x_j \geq 0 \qquad \text{for all investments } j \end{aligned}$$

λ is the risk parameter.

MAD Markowitz Problem

$$
\begin{aligned}
\text{maximize} \quad & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right| \\
\text{subject to} \quad & \sum_j x_j = 1 \\
& x_j \geq 0 \qquad \text{for all investments } j
\end{aligned}
$$

Not a linear programming problem. But it's easy to convert.

Stochastic Dominance

Given: two random variables, V and S.

- **First order stochastic dominance**: $V \succeq_1 S$ means

$$F_V(\eta) \le F_S(\eta), \qquad \text{for all } \eta \in \mathbb{R},$$

where F_V and F_S denote the cummulative distribution functions of V and S, respectively:

$$F_V(\eta) = \mathbb{P}\{V \le \eta\} \qquad \text{for } \eta \in \mathbb{R}.$$

- **Second order stochastic dominance**: $V \succeq_2 S$ means

$$\int_{-\infty}^{\eta} F_V(\xi) d\xi \le \int_{-\infty}^{\eta} F_S(\xi) d\xi, \qquad \text{for all } \eta \in \mathbb{R}.$$

Henceforth, the term *stochastic dominance* without qualifier will refer to *second order* dominance.

Stochastic Dominance and Utility Theory

Second order stochastic dominance characterizes those random variables that every risk averse decision maker would prefer to a given random variable:

Theorem *Random variable V stochastically dominates random variable S if and only if $\mathbb{E}(U(V)) \geq \mathbb{E}(U(S))$ for every increasing concave function $U(\cdot)$.*

Theorem *There are optimal solutions to the quadratic Markowitz model that are stochastically dominated by other (non-optimal) portfolios.*

Theorem *In the MAD Markowitz model, optimal portfolios are not stochastically dominated at least for all $\lambda \geq 2$.*

MAD Markowitz: LP Formulation

$$
\begin{array}{llll}
\text{maximize} & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t & & \\
\text{subject to} & -y_t \le \sum_j D_{tj} x_j \le y_t & & \text{for all times } t \\
& \sum_j x_j = 1 & & \\
& x_j \ge 0 & & \text{for all investments } j \\
& y_t \ge 0 & & \text{for all times } t
\end{array}
$$

Adding Slack Variables w_t^+ and w_t^-

$$
\begin{array}{lrll}
\text{maximize} & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t & & \\
\text{subject to} & -y_t - \sum_j D_{tj} x_j + w_t^- = 0 & & \text{for all times } t \\
 & -y_t + \sum_j D_{tj} x_j + w_t^+ = 0 & & \text{for all times } t \\
 & \sum_j x_j = 1 & & \\
 & x_j \geq 0 & & \text{for all investments } j \\
 & y_t, w_t^-, w_t^+ \geq 0 & & \text{for all times } t
\end{array}
$$

The Solution for Large λ

Varying the risk parameter $0 \leq \lambda < \infty$ produces the *efficient frontier*.

Large values of λ favor reward whereas small values favor minimizing risk.

Beyond some finite threshold value for λ, the optimal solution will be a portfolio consisting of just one asset—the asset j^* with the largest average return:

$$\mu_{j^*} \geq \mu_j \qquad \text{for all } j.$$

It's easy to identify basic vs. nonbasic variables:

- Variable x_{j^*} is basic whereas the remaining x_j's are nonbasic.
- All of the y_t's are basic.
- If $D_{tj^*} > 0$, then w_t^- is basic and w_t^+ is nonbasic. Otherwise, it is switched.

The Basic Optimal Solution for Large λ

Let

$$T^+ = \{t : D_{tj^*} > 0\}, \qquad T^- = \{t : D_{tj^*} < 0\}, \qquad \text{and} \qquad \epsilon_t = \begin{cases} 1, & \text{for } t \in T^+ \\ -1, & \text{for } t \in T^- \end{cases}$$

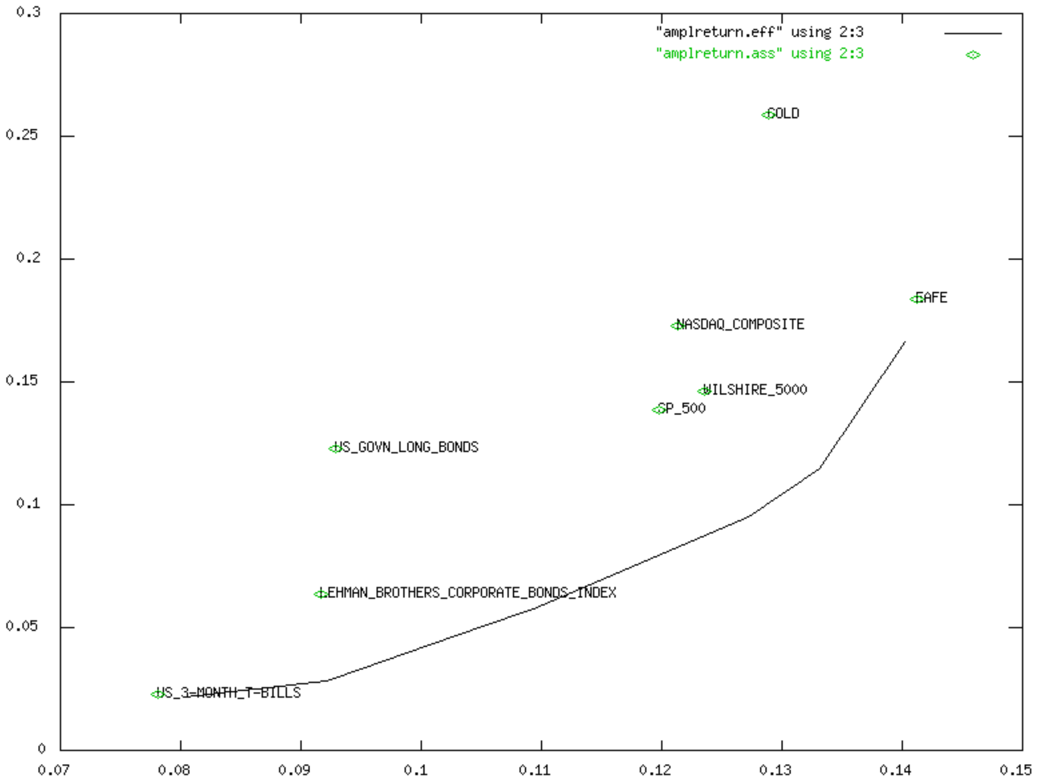
It's tedious, but here's the optimal dictionary:

$$\begin{array}{rcllllr}
\zeta & = & \frac{1}{T}\sum_{t=1}^{T} \epsilon_t D_{tj^*} & -\frac{1}{T}\sum_{j\neq j^*}\sum_{t=1}^{T}\epsilon_t (D_{tj} - D_{tj^*})x_j & -\frac{1}{T}\sum_{t\in T^-} w_t^- & -\frac{1}{T}\sum_{t\in T^+} w_t^+ & \\
& & +\lambda\mu_{j^*} & +\lambda\sum_{j\neq j^*}(\mu_j - \mu_{j^*})x_j & & & \\
\hline
y_t & = & -D_{tj^*} & -\sum_{j\neq j^*}(D_{tj} - D_{tj^*})x_j & +w_t^- & & t \in T^- \\
w_t^- & = & 2D_{tj^*} & +2\sum_{j\neq j^*}(D_{tj} - D_{tj^*})x_j & & +w_t^+ & t \in T^+ \\
y_t & = & D_{tj^*} & +\sum_{j\neq j^*}(D_{tj} - D_{tj^*})x_j & & +w_t^+ & t \in T^+ \\
w_t^+ & = & -2D_{tj^*} & -2\sum_{j\neq j^*}(D_{tj} - D_{tj^*})x_j & +w_t^- & & t \in T^- \\
x_{j^*} & = & 1 & -\sum_{j\neq j^*} x_j & & &
\end{array}$$

Efficient Frontier

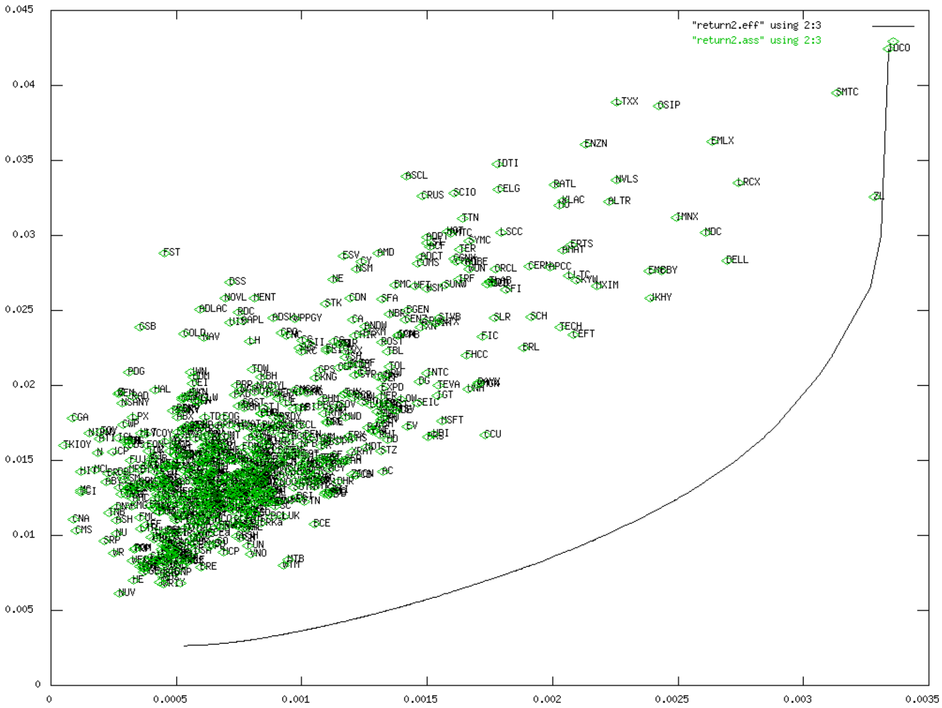
Varying risk parameter λ produces the so-called *efficient frontier*.

λ	US 3-Month T-Bills	S&P 500	Lehman Bros. Corp. Bonds	NASDAQ Comp.	Wilshire 5000	Gold	EAFE	Reward (pct)	Risk
17.872							100.0	14.1	0.184
6.236						8.0	92.0	14.0	0.166
2.555					25.6	29.2	45.1	13.3	0.115
1.863			1.2		51.3	35.4	12.1	10.9	0.058
1.788		2.8	58.7			18.4	20.1	10.9	0.057
1.778			61.4			17.4	21.2	10.3	0.047
1.609	22.3		45.6			13.5	18.6	9.3	0.029
1.417	51.5		29.6		2.1	7.2	9.6	9.2	0.029
0.180	55.0		26.8	1.3		6.5	10.4	8.0	0.022
0.000	92.6		4.7			0.9	1.8	7.9	0.021



"amplreturn.eff" using 2:3
"amplreturn.ass" using 2:3
GOLD
EAFE
NASDAQ_COMPOSITE
WILSHIRE_5000
SP_500
US_GOVN_LONG_BONDS
LEHMAN_BROTHERS_CORPORATE_BONDS_INDEX
US_3-MONTH_T-BILLS
0.3
0.25
0.2
0.15
0.1
0.05
0
0.07
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15

Daily Returns for 12 Years on 719 Assets



Click here for an expanded browser view.

Computing the Efficient Frontier

Using a reasonably efficient code for the *parametric simplex method* (simpo), it took *22,000* pivots and *1.5 hours* to solve for *one point* on the efficient frontier.

Customizing this same code to solve parametrically for every point on the efficient frontier, it took *23,446* pivots and *57 minutes* to compute *every point* on the frontier.

The efficient frontier consists of *23,446* distinct portfolios. Click here for a partial list (*warning: the file is 2.5 MBytes*). The complete list makes a 37 MByte file.

Upside Risk—An Oxymoron

$$\rho_2(x) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j D_{tj}x_j\right)^2 \quad \Rightarrow \quad \rho_2^-(x) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j D_{tj}x_j\right)_-^2$$

$$\rho_1(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j D_{tj}x_j\right| \quad \Rightarrow \quad \rho_1^-(x) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j D_{tj}x_j\right)_-$$

where $(x)_- = \min(x, 0)$.

Theorem *(Trivial '02)* $\rho_1^-(x) = \rho_1(x)/2$.

Corollary *Efficient frontier is "good" for all* $\lambda \geq 1$.

Note: No analogous result for semi-variance.

Means, Medians, and Quantiles

Mean is solution to: $\min_z \sum_j \left(b_j - z\right)^2$

Median is solution to: $\min_z \sum_j \left|b_j - z\right|$

In MAD portfolio model, replace

$$\left|\sum_j R_j(t)x_j - \mu(x)\right| \text{ with } \left|\sum_j R_j(t)x_j - z\right|$$

to get mean absolute deviation from the median.

Theorem *Efficient frontier is "good" for all $\lambda \geq 1$.*

REVIEW

- A portfolio is *bad* if another portfolio dominates it (stochastically).
- Many portfolios on Markowitz's "efficient frontier" are bad.
- MAD Markowitz isn't bad.
- MAD Markowitz is a parametric LP.
- Even more, using the parametric simplex method the entire efficient frontier can be computed in the time normally required to find just one point on the frontier.
- Lastly, our efficient frontier is completely determined by a finite set of portfolios (vs. a continuum).

Paper was published a year ago in *Econometrica*.

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