The problem is to compute the correct velocity vector to impart to a golf ball on a green in order that the ball will roll to the cup and fall in. It should arrive at the cup with as little remaining speed as possible. Let $u(t) = (x(t), y(t), z(t))$ denote the position of the ball at time t. Let $v(t) = (v_x(t), v_y(t), v_z(t))$ denote its velocity and $a(t) = (a_x(t), a_y(t), a_z(t))$ its acceleration.

We suppose that the elevation of the green is given by a function $z(x, y)$. That is, the surface of the green is given as $(x, y, z(x, y))$. Therefore, two tangent vectors are $(1, 0, \frac{\partial z}{\partial x})$ and $(0, 1, \frac{\partial z}{\partial y})$. By taking the cross product of these two vectors, we obtain an upward pointing normal vector to the surface:

$$(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1).$$

The normal force N exerted by the surface of the green on the golf ball points in this direction. We must work out its magnitude. Equivalently, we must determine N_z in

$$N = (N_x, N_y, N_z) = (-\frac{\partial z}{\partial x}N_z, -\frac{\partial z}{\partial y}N_z, N_z).$$

For planar surfaces, the magnitude of N must be such that the total force in the normal direction vanishes (to keep the ball rolling on the flat surface). But for curved surfaces, the magnitude of N must provide the oomph necessary to get the ball to follow the curvature of the green (the so-called g-force). Specifically, we must have that the total force in the normal direction equates to the mass times the component of acceleration in this direction:

$$\|N\| - mg\frac{e_z \cdot N}{\|N\|} = m\frac{a(t) \cdot N}{\|N\|},$$

where m is the mass of the ball, g is the acceleration due to gravity, and e_z is the unit vector pointing in the positive z-direction. From this relation we can deduce that

$$N_z = m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$$

and, of course, that

$$N_x = -\frac{\partial z}{\partial x}N_z \qquad N_y = -\frac{\partial z}{\partial y}N_z.$$

In addition to the force of gravity and the normal force imparted by the ground, we assume there is friction which is taken to be proportional to the normal force and to point in a direction opposite to the velocity:

$$F = -\mu\|N\|\frac{v}{\|v\|}.$$

The equations of motion can now be stated precisely:

$$\begin{aligned} v &= \dot{u} \\ a &= \dot{v} \\ ma &= N + F - mge_z. \end{aligned}$$

Of course, if the equations of motion are satisfied in the x and y directions then they will automatically be satisfied in the z direction too.

The initial and final positions are known: $u(0) = u_0$ and $u(T) = u_f$. But, the time T at which the final position is reached is not known a priori.

In general, the solution to the problem is not unique. One may hit the ball hard and directly toward the cup or one may hit the ball gently but then must pick the initial direction carefully as the trajectory of the ball will bend according to the contours of the surface. To get a unique solution we can say minimize the speed as the ball falls in the cup:

$$\text{minimize } v_x(t)^2 + v_y(T)^2.$$