THE FALLING SLINKY

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Abstract. Consider a slinky that is held at one end while the rest of it is allowed to hang freely. If the slinky is released, then as it falls due to gravity it also shrinks in extent due to the contracting force in the slinky itself. A few years ago it was discovered that the net effect of these two changes exactly counteract each other at the bottom extent of the slinky and therefore the bottom of the slinky remains utterly motionless until the full slinky is complete compressed at which point the bottom edge begins to loose height as the whole thing falls together. In this paper, we describe a simple model that closely approximates the dynamics of a real slinky and we give a careful mathematical explanation for this phenomenon.

1. Introduction. In this note, we study a simple dynamical system that is an approximation to a slinky that falls after being released from a vertically hanging position. We model the slinky as a chain of $n+1$ bodies each having mass m. Suppose that each adjacent pair of bodies is connect by a spring having spring constant k. The fact that the bottom of the slinky remains utterly motionless until the top of the slinky reaches it was first pointed out by M.G. Calkin [1] and 1993. In 2000, Martin Gardner [3] described the unexpected behavior in one of his monthly colums. Soon thereafter, it was investigated/studied by others; see, e.g., [4, 5, 6, 2].

In this paper, we present a detailed analysis of both the discrete n-body approximation to the problem as well as the continuum limit of that approximation. The only physics prerequisite needed to understand this paper is basic knowledge of Newton's second law of dynamics ($F = ma$). The paper is written for mathematicians (and students of mathematics) who wish to see how elementary calculus and algebra can be used to analyze cool physics problems such as the falling slinky.

2. Differential Equation. We are interested in the interplay between the effects of gravity and the effects of the springs. In particular, we imagine holding body "0" and letting the rest of them hang below pulled down by gravity but held by the springs. Let

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

denote the vertical heights of the $n+1$ masses. Writing down the force balance equation at each mass, we arrive at the following system of differential equations:

$$m\ddot{y} + kAy = -mge, \tag{2.1}$$

where g denotes the acceleration due to gravity, e denotes the vector of all ones, and

$$A = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

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3. Initial Conditions. Suppose that initially the masses are hanging in such a way that the downward force due to gravity exactly matches the upward force due to the springs so that the "slinky" remains stationary—at least until we let go of body 0. Let's pick our coordinate system so that

$$y_0(0) = 0.$$

Setting the acceleration equal to zero for each of the other bodies, we get

$$\begin{aligned} y_{j-1}(0) - 2y_j(0) + y_{j+1}(0) &= \frac{mg}{k}, \qquad j = 1, 2, \ldots, n-1 \\ y_{n-1}(0) - y_n(0) &= \frac{mg}{k}. \end{aligned}$$

To find an explicit formula for the initial solution, we look for a solution of the form $y_j(0) = aj + bj^2$. Plugging this guess into the equations above, it is easy to deduce that

$$b = \frac{mg}{2k} \qquad \text{and} \qquad a = -\frac{mg}{k}(n + 1/2).$$

Hence,

$$y_j(0) = -\frac{mg}{k}\left(nj - \frac{1}{2}j(j-1)\right), \qquad j = 0, 1, 2, \ldots, n. \tag{3.1}$$

4. Particular Solution. If we let all $n+1$ masses start together at the origin, then there are no spring forces and all masses fall downward due to gravity. Hence, we propose that

$$y_{\mathrm{P}} = -\frac{gt^2}{2}e$$

is a particular solution to the differential equation. Indeed, it is easy to see that

$$\ddot{y}_{\mathrm{P}} = -ge$$

and that

$$Ae = 0.$$

Hence,

$$m\ddot{y}_{\mathrm{P}} + kAy_{\mathrm{P}} = -mge$$

as required.

5. Homogeneous Solution. The homogeneous equation is given by

$$m\ddot{y} + kAy = 0.$$

We look for solutions of the form

$$y = e^{Bt}c$$

for some symmetric matrix B and some vector c. Differentiating twice, we get

$$\ddot{y} = B^2e^{Bt}c$$

and so

$$m\ddot{y} + kAy = mB^2 y + kAy = 0.$$

This equation is satisfied if we pick

$$B = \pm\sqrt{\frac{k}{m}}\sqrt{A}i$$

where $\sqrt{A}$ denotes the positive semidefinite square root of the positive semidefinite matrix A and i denotes, as usual, the square root of -1. Hence, the general solution to the homogeneous equation can be written as

$$y_{\mathrm{H}} = e^{\sqrt{k/m}\sqrt{A}it}c_+ + e^{-\sqrt{k/m}\sqrt{A}it}c_-$$

and the general solution to the original equation is the sum of a particular solution and the homogeneous solution:

$$y = e^{\sqrt{k/m}\sqrt{A}it}c_+ + e^{-\sqrt{k/m}\sqrt{A}it}c_- - \frac{gt^2}{2}e.$$

6. Solution with Given Initial Conditions. To find the solution to our hanging slinky that's been released at time $t = 0$, we just need to use the initial conditions to solve for c_+ and c_-. Plugging $t = 0$ into our solution, we get

$$y(0) = c_+ + c_-.$$

And, using the fact that the velocity vector vanishes at $t = 0$, we get that

$$\dot{y}(0) = \sqrt{\frac{k}{m}}\,\sqrt{A}\,ic_+ - \sqrt{\frac{k}{m}}\,\sqrt{A}\,ic_- = 0$$

from which we deduce that $c_+ = c_-$. Putting these facts together, we get that

$$c_+ = c_- = \frac{y(0)}{2}$$

and so

$$\begin{aligned} y(t) &= \frac{e^{\sqrt{k/m}\sqrt{A}\,it} + e^{-\sqrt{k/m}\sqrt{A}\,it}}{2}y(0) - \frac{gt^2}{2}e \\ &= \cos\left(\sqrt{k/m}\,\sqrt{A}\,t\right)y(0) - \frac{gt^2}{2}e \\ &= \left(I - \frac{k}{m}\,\frac{t^2}{2!}\,A + \frac{k^2}{m^2}\,\frac{t^4}{4!}\,A^2 - + \cdots\right)y(0) - \frac{gt^2}{2}e. \end{aligned}$$

Our main interest is in the position of the bottom-most mass, y_n, as a function of time:

$$y_n(t) = e_n^T y(t)$$

where e_n denotes the $n + 1$ vector that is all zeros except for a one in the last position.

Now, if we refer back to the equations that defined our initial conditions, we see that all but the 0^{th} element of $Ay(0)$ is equal to $-mg/k$ and the 0^{th} element turns out to be mgn/k. So,

$$Ay(0) = -\frac{mg}{k}e + \frac{mg}{k}\,(n+1)\,e_0$$

where e_0 denotes the $n+1$ vector that is one in the 0^{th} position and zero elsewhere. From this it follows that

$$e_n A y(0) = -\frac{mg}{k}$$

Hence, the quadratic term in the Taylor series expansion of the cosine exactly cancels the quadratic acceleration due to gravity term.

Finally, let us consider the other terms in the Taylor series expansion. Since the row sums of the A matrix all vanish, it follows, as mentioned before, that $Ae = 0$. Hence, for $j \geq 1$,

$$A^{j+1} y(0) = \frac{mg}{k}\ (n+1)\ A^j e_0.$$

It is easy to see that Ae_0 is all zeros except for the first two elements. Similarly, $A^2 e_0$ is all zeros except for the first three elements. And, by induction, it is easy to check that $A^j e_0$ is zero in all by the first $j+1$ elements. Hence, the last, i.e. $(n+1)$-st, element remains zero until $j = n$. Therefore the last component of the vector vanishes in the Taylor series until the t^{2n} term. Hence, the motion of the bottom mass in the chain appears to remain fixed until time is sufficiently great that the t^{2n}-th term becomes significant.

7. Continuum Solution: The Limit as $n \longrightarrow \infty$. Now let's study the limit as n tends to infinity. Before taking limits, we need to scale things appropriately. First off, rather than having each body have mass m, we will assume that each body has mass m/n so that the total mass is roughly m. Also, we need to scale the spring constant appropriately. It turns out that the correct choice there is to increase the spring constant in direct proportion to the number of masses: kn instead of k. With these rescalings, the differential equation (2.1) becomes

$$\ddot{y} + \frac{k}{m}\frac{A}{1/n^2} y = -ge. \tag{7.1}$$

We also change our indexing from a simple index j that counts the bodies to a real number x that represents the fraction of the way from the top of the slinky to the bottom. So, $x = 0$ represents the top, $x = 1$ represents the bottom in general the index j is replaced by $x = j/n$. With this notation, all rows of the matrix $A/(1/n^2)$ except the first and the last converge to the negative of the second derivative of height y with respect to the variable x:

$$\frac{A}{1/n^2} \longrightarrow \frac{\partial^2}{\partial x^2}.$$

The first row (corresponding to $x = 0$) and the last row (corresponding to $x = 1$) require a different scaling. Rather than $1/n^2$ in the denominator, we need $1/n$. So, for these two cases, we must divide (7.1) by n before taking limits. Doing this and taking limits, we see that the first term on the left and the right-hand side both vanish and so we are left with:

$$\frac{\partial y}{\partial x}(x,t) = 0, \qquad \text{for } x \in \{0,1\} \text{ and } t \geq 0.$$

In the continuum limit the initial conditions given by (3.1) become

$$y(x,0) = -\frac{mg}{k}\left(x - \frac{1}{2}x^2\right).$$

To summarize, the differential equation that defines the continuum problem is a particular solution to the *wave equation*:

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2}(x,t) - \frac{k}{m}\frac{\partial^2 y}{\partial x^2}(x,t) &= -g, && 0 < x < 1,\ t > 0, \\
y(x,0) &= -\frac{mg}{k}\left(x - \frac{1}{2}x^2\right), && 0 \le x \le 1, \\
\frac{\partial y}{\partial t}(x,0) &= 0, && 0 \le x \le 1, \\
\frac{\partial y}{\partial x}(x,t) &= 0, && x \in \{0,1\},\ t \ge 0.
\end{aligned}
$$

To solve this wave equation, we must find a particular solution to the differential equation and then the most general solution to the associated homogeneous solution.

A particular solution is easy to produce:

$$y_{\mathrm{P}}(x,t) = -\frac{1}{2}gt^2.$$

Checking that this is a solution is trivial. The general solution to the homogeneous equation is also easy to write down:

$$y_{\mathrm{H}}(x,t) = f_{\pm}\left(x \pm \sqrt{k/m}\, t\right),$$

where $f_{\pm}$ are arbitrary functions. Again, it is trivial to check that such a function satisfies the homogeneous wave equation:

$$\frac{\partial^2 y_{\mathrm{H}}}{\partial t^2}(x,t) - \frac{k}{m}\frac{\partial^2 y_{\mathrm{H}}}{\partial x^2}(x,t) = 0$$

So, the general solution to the nonhomogeneous equation is

$$y(x,t) = f_{+}\left(x + \sqrt{k/m}\, t\right) + f_{-}\left(x - \sqrt{k/m}\, t\right) - \frac{1}{2}gt^2.$$

All that remains is to use the boundary conditions to discover the exact form of f_+ and f_-. From the boundary conditions at $t = 0$, we see that

$$y(x,0) = f_+(x) + f_-(x) = -\frac{mg}{k}x\left(1 - \frac{x}{2}\right), \qquad 0 < x < 1,$$

and

$$\frac{\partial y}{\partial t}(x,0) = \frac{k}{m}f_+(x) - \frac{k}{m}f_-(x) = 0, \qquad 0 < x < 1.$$

From this latter condition we deduce that $f_+(x) = f_-(x)$ for $0 < x < 1$ and plugging this into the first condition we get

$$f_+(x) = f_-(x) = -\frac{mg}{2k}x\left(1 - \frac{x}{2}\right), \qquad 0 < x < 1.$$

Finally, from the boundary conditions at $x = 0$ and $x = 1$, we get after a trivial change of variables:

$$f'(t) = -f'(-t) \quad \text{and} \quad f'(1+t) = -f'(1-t), \qquad \text{for } t > 0$$

(note: we have dropped the subscript $\pm$ since the two functions are the same). These last conditions tell us that we can extend the formula for f beyond the interval $[0, 1]$ by successive reflection operations. It is now easy to check that f is a periodic function with period 2 defined by

$$f(x) = -\frac{mg}{4k} x\,(2-x)\,, \qquad 0 \le x \le 2.$$

In other words, the general formula for f is

$$f(x) = -\frac{mg}{4k}\,(x \bmod 2)\,(2 - x \bmod 2)\,.$$

8. The Temporal Evolution of the Bottom of the Slinky. Finally, we can analyze the motion of the "bottom" of the slinky. The bottom is given by $x = 1$. So, we have

$$y(1,t) = f(1 + \sqrt{k/m}\,t) + f(1 - \sqrt{k/m}\,t) - \frac{1}{2}gt^2.$$

For $0 \le t \le \sqrt{m/k}$, both arguments to the function f in the formula for y are in the main interval $[0, 2]$ and therefore the formula is quite simple in this case:

$$\begin{aligned} y(1,t) &= -\frac{mg}{4k}\left(1+\sqrt{\frac{k}{m}}\,t\right)\left(1-\sqrt{\frac{k}{m}}\,t\right) - \frac{mg}{4k}\left(1-\sqrt{\frac{k}{m}}\,t\right)\left(1+\sqrt{\frac{k}{m}}\,t\right) - \frac{1}{2}gt^2 \\ &= -\frac{mg}{2k}\left(1 - \frac{k}{m}\,t^2\right) - \frac{1}{2}gt^2 \\ &= -\frac{mg}{2k}. \end{aligned}$$

To summarize: we have shown that the bottom of the slinky remains utterly motionless during the time interval $0 \le t \le \sqrt{m/k}$.

A plot of $y(x,t)$ versus t, for some choices of x between zero and one is shown in Figure 8.1.

9. The Realistic Case: Inelasitc Collisions. It is clear from Figure 8.1 that the bodies can freely pass through each other as the chain of bodies falls. A real slinky is not "porous" like this. Instead, we should assume that the balls can't pass through each other. This version is hard to solve analytically. But, it's not hard to simulate such a motion. A Matlab program that computes the path of the balls is shown in Figure 9.1. The output of this Matlab code is shown in Figure 9.2.

From these numerical computations, it is clear that the each body in the non-porous chain remains stationary until the collapsed bodies from above collide with it. Based on this observation, we can derive a formula for the position of the top of the continuum slinky as a function of time. Let $\mu(t)$ denote the center of mass of the system at time t. For time $t = 0$, we compute the center of mass to be

$$\mu(0) \;=\; \int_0^1 y(x,0)dx \;=\; -\frac{mg}{k}\int_0^1 \left(x - \frac{1}{2}x^2\right)dx \;=\; -\frac{1}{3}\frac{mg}{k}. \tag{9.1}$$

Once released, the center of mass of the slinky accelerates downward at rate g so, at time t, we have

$$\mu(t) = \mu(0) - \frac{1}{2}gt^2 = -\frac{1}{3}\frac{mg}{k} - \frac{1}{2}gt^2.$$

We now assume that a certain proportion of the top of the slinky has collapsed to the bottom position of that proportion. Let x_t denote this proportion at time t. So, we have

$$y(x,t) = \begin{cases} y(x_t, 0), & x \le x_t, \\ y(x,0), & x \ge x_t. \end{cases}$$

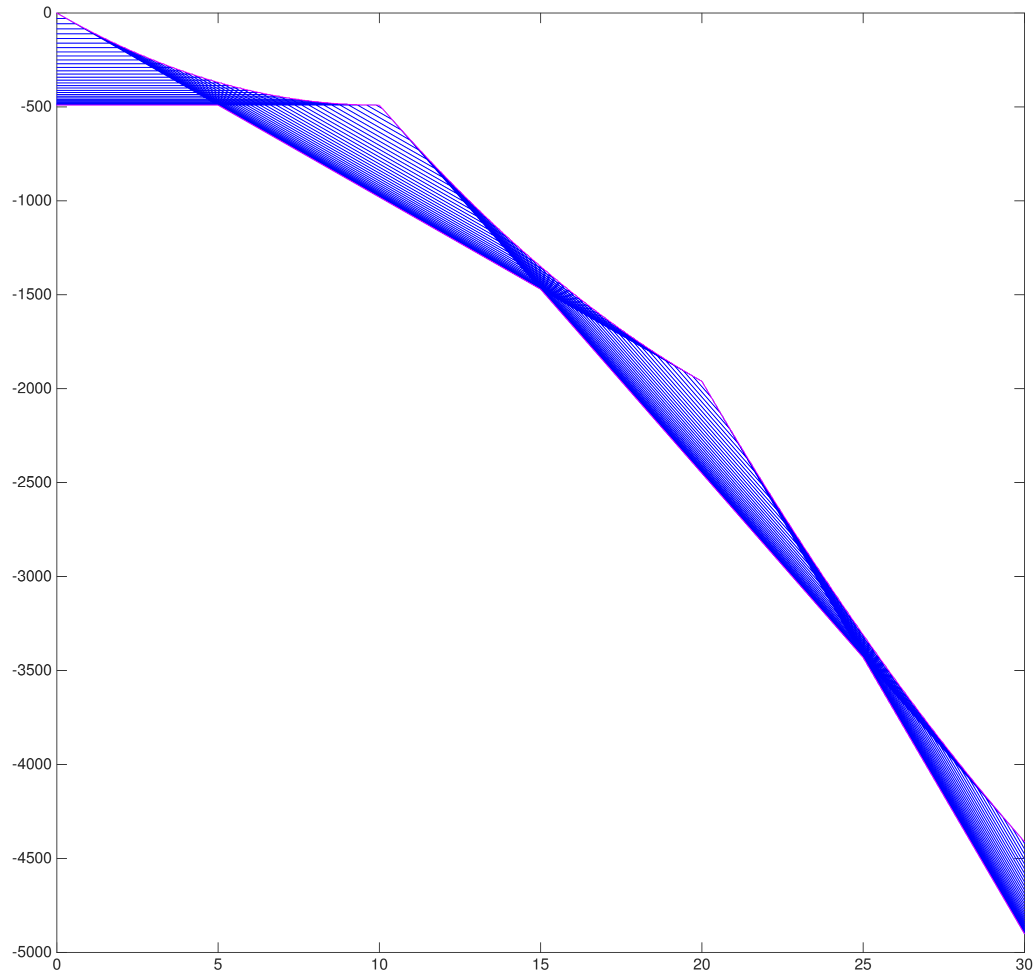

FIG. 8.1. *A plot of $y(x,t)$ versus t for $x = 0, 1/10, 2/10, \ldots, 1$.*

and we can compute the center of mass of this partially collapsed slinky:

$$\begin{aligned}\mu(t) &= \int_0^1 y(x,t)dx \\ &= \int_0^{x_t} y(x_t,0)dx + \int_{x_t}^1 y(x,0)dx \\ &= x_t y(x_t,0) - \frac{mg}{k}\int_{x_t}^1 \left(x - \frac{x_t^2}{2}\right)dx\end{aligned} \tag{9.2}$$

$$= -x_t \frac{mg}{k}\left(x_t - \frac{x_t^2}{2}\right) - \frac{mg}{k}\left(\frac{1}{3} - \frac{x_t^2}{2} + \frac{x_t^3}{6}\right)$$

Equating the right-hand sides of (9.1) and (9.2) and simplifying, we see that x_t must satisfy a cubic equation:

$$\frac{x_t^3}{3} - \frac{x_t^2}{2} + \frac{1}{2}\frac{k}{m}t^2 \;=\; 0.$$

The general solution to this equation is

$$x_t = \frac{1}{2} - uC_t - \frac{1}{4uC_t},$$

where u is any of the three complex roots of unity and

$$C_t = \left(\frac{-\frac{1}{4} + \frac{3}{2}\frac{k}{m}t^2 + \sqrt{\frac{3}{4}\frac{k}{m}t^2\left(-1 + 3\frac{k}{m}t^2\right)}}{2}\right)^{1/3}$$

We need to pick the correct root of unit. It must be pick so as to ensure that x_t lies between 0 and 1. Careful analysis of the case where t is small allows us to determine which of the three roots of unity is the correct one.

We start by considering the case $t = 0$. In this case, the cubic equation is easy:

$$\frac{x_0^3}{3} - \frac{x_0^2}{2} = 0 \qquad \Longrightarrow \qquad x_0 = 0,\ 0,\ 3/2.$$

Let us try to determine which cube root of unity is associated with $3/2$. For $t = 0$, we have

$$C_0 = -1/2$$

and so we get

$$x_0 \;=\; \frac{1}{2}\left(1 + u + 1/u\right).$$

Clearly, the root $u = 1$ yields the $3/2$ solution and the other two roots, $u_\pm = e^{\pm 2\pi i/3}$, both yield a zero solution. To determine which of these two roots is the correct one, we consider the case where $\sqrt{k/m}\,t$ is tiny, say ε. In this case, we have

$$C_t \;\approx\; \left(-\frac{1}{8} + \frac{\sqrt{3}}{4}\varepsilon i\right)^{1/3} \;\approx\; -\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}\,i$$

Plugging into our equation for x_t, we get

$$\begin{aligned}
x_t^\pm &\approx \frac{1}{2} - e^{\pm 2\pi i/3}\left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}\,i\right) - \frac{1}{4e^{\pm 2\pi i/3}\left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}\,i\right)} \\
&= \frac{1}{2} + e^{\pm 2\pi i/3}\left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}\,i\right) + e^{\mp 2\pi i/3}\frac{1}{2 - \frac{4\varepsilon}{\sqrt{3}}\,i} \\
&= \frac{1}{2}\left(1 + e^{\pm 2\pi i/3}\left(1 - \frac{2\varepsilon}{\sqrt{3}}\,i\right) + e^{\mp 2\pi i/3}\frac{1}{1 - \frac{2\varepsilon}{\sqrt{3}}\,i}\right) \\
&\approx \frac{1}{2}\left(1 + e^{\pm 2\pi i/3}\left(1 - \frac{2\varepsilon}{\sqrt{3}}\,i\right) + e^{\mp 2\pi i/3}\left(1 + \frac{2\varepsilon}{\sqrt{3}}\,i\right)\right) \\
&= \frac{\varepsilon}{\sqrt{3}}i\left(-e^{\pm 2\pi i/3} + e^{\mp 2\pi i/3}\right) \\
&= \pm\varepsilon.
\end{aligned}$$

From this computation, we see that u_+ produces x_t values that are positive for small t whereas u_- produces negative values. Hence, u_+ is the correct root.

Finally, we note that as a bi-product of this analysis we can compute how long it takes for the slinky to fully collapse. The moment of full collapse corresponds to $x_t = 1$. Hence, from our cubic equation, we get that

$$-\frac{1}{6} + \frac{1}{2}\frac{k}{m}t^2 = 0.$$

Solving for t we get

$$t = \sqrt{\frac{m}{3k}}.$$

Figure 9.2 shows the output produced by the Matlab code shown in Figure 9.1. The red curve plots the center of mass as a function of time.

Finally, we note that a WebGL online integration of the differential equation can be accessed here:

`http://www.princeton.edu/~rvdb/WebGL/Slinky.html`

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```
n = 50;
dt = 0.001;
endtime = 10;
iters = round(endtime/dt);

k = 1*n;
m = 1/n;
g = 9.8;

A = full(gallery('tridiag',n+1,-1,2,-1));
A(1,1) = 1;
A(end,end) = 1;
e = ones(n+1,1);

t = zeros(n+1,1);

j = (0:n)';
y = -(m*g/k)*(n*j - 0.5*j.*(j-1));
ydot = zeros(n+1,1);
ydot2 = -(k/m)*A*y - g*e;

ylim([1.1*min(y) 1.1*max(y)]);
xlabel('time');
ylabel('y');
hold on;

for iter = 0:iters
    ydot = ydot + ydot2*dt/2;
    y = y + ydot*dt/2;

    for i = 1:n
        for j = (i+1):(n+1)
          if y(j) < y(i)
break;
  end
end
j = j-1;
ybar = mean(y(i:j));
ydotbar = mean(ydot(i:j));
y(i:j) = ybar;
ydot(i:j) = ydotbar;
    end

    y = y + ydot*dt/2;
    ydot2 = -(k/m)*A*y - g*e;
    ydot = ydot + ydot2*dt/2;
    t = t + dt;
    if mod(iter,10) == 0
        TT(:,1+iter/10) = t(1:10:end);
        YY(:,1+iter/10) = y(1:10:end);
    end
end
plot(TT',YY','b-');
hold on;
plot(mean(TT),mean(YY),'r-');
hold off;
```

FIG. 9.1. *Matlab code for n nonporous balls.*

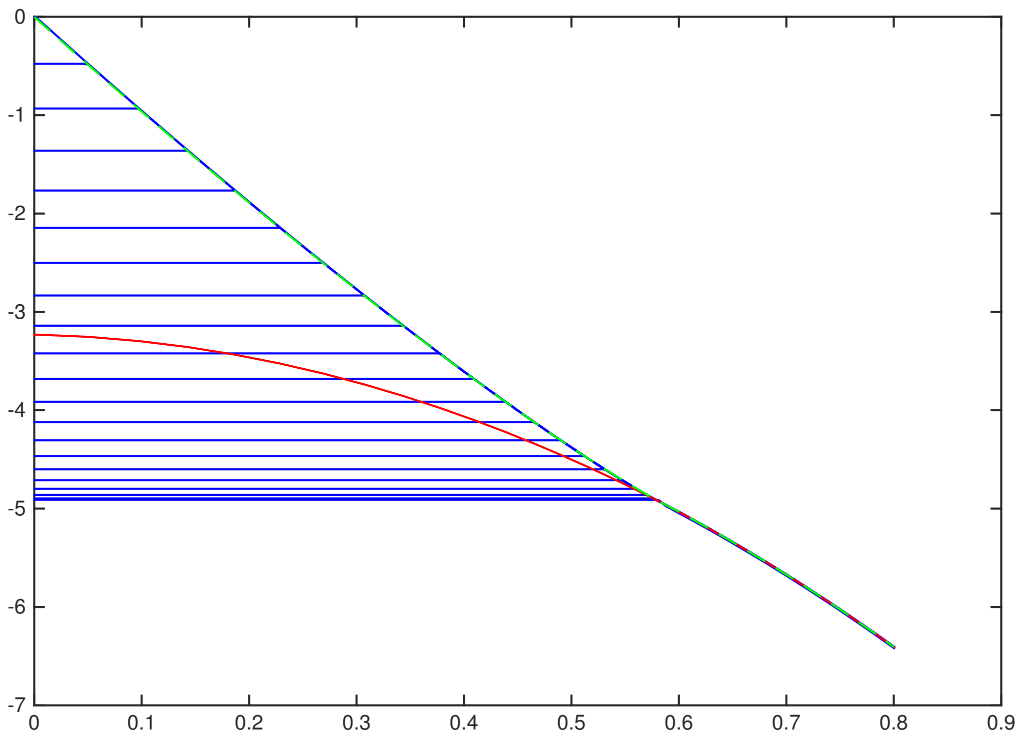

FIG. 9.2. *A plot of $y(x,t)$ versus t for each of 500 non-porous balls. The motion over time of every 10th ball is shown. The dashed cyan line shown along the top edge was computed from the solution to the solid continuum model ($y(x_t, 0)$).*