(1) *Inverse Dirichlet Problem.* Consider a thin square metal plate, which for simplicity we shall model as a unit square in $\mathbb{R}^2$. If the temperature distribution at the boundary of the plate is held fixed over time and the temperature distribution throughout the interior of the plate is allowed to equilibrate, then the temperate $u(x, y)$ at an interior point (x, y) satisfies the following partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(such a function u is called *harmonic*). Also, if $f(x, y)$ denotes the fixed temperature at the boundary point (x, y), then u also satisfies

$$u(x, y) = f(x, y)$$

on the boundary. The Dirichlet problem is the problem of finding a harmonic function u in a given domain (in our case the square) having given boundary values f. For the *inverse Dirichlet problem*, the function f is assumed to be unknown and one would like to estimate it given a finite number of measurements of u in the interior of the square. If we are willing to discretize the square, we can use linear programming to tackle this problem. Indeed, suppose that the square is replaced by a two dimensional integer lattice: $\{(i, j) : i = 0, 1, \ldots, M, j = 0, 1, \ldots, M\}$. Then, u is defined only at the $(M + 1)^2$ lattice points and the partial differential equation is replaced by the following difference equation:

(1) $$u(i, j) - (u(i, j + 1) + u(i, j - 1) + u(i + 1, j) + u(i - 1, j))/4 = 0,$$

for $i = 1, 2, \ldots, M - 1$, and $j = 1, 2, \ldots, M - 1$. Suppose that measurements are taken at N points,

$$(i_1, j_1), (i_2, j_2), \ldots, (i_N, j_N),$$

and that the kth measurement is denoted by d_k. Then, to be consistent with the data, our solution must satisfy the following N equations:

(2) $$u(i_k, j_k) = d_k, \qquad k = 1, 2, \ldots, N.$$

Any solution to (1) and (2) gives a solution to the discrete inverse Dirichlet problem. In general, there are many such solutions since these systems involve $(M - 1)^2 + N$ equations in $(M + 1)^2$ unknowns (we assume here that $4M > N$). To help select a single solution from the many, suppose we introduce one extra variable, $u_{\max}$, add the stipulation that

$$0 \le u(i, j) \le u_{\max},$$

for $i = 1, 2, \ldots, M - 1$, $j = 1, 2, \ldots, M - 1$, and then minimize $u_{\max}$. The resulting problem is a linear programming problem.

(a) Write down the linear program.

(b) How many nonbasic variables are there?

Assume now that this problem has been solved using the simplex method and the optimal solution has $u_{\max} > 0$.

(c) Using (1), show that no interior value can be at the lower bound, zero, or at the upper bound, $u_{\max}$.

(d) How many of the $4M$ boundary values are not at the lower or upper bound? What can you say about the nature of the solution as M tends to infinity?