Newton & Kepler

Freshman Seminar 131

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Orbital Mechanics

Here's a link to my Sky & Telescope article: Measuring the Astronomical Unit

Isaac Newton:

Newton's Second Law of Motion:

$$F = ma$$

Newton's Law of Gravity:

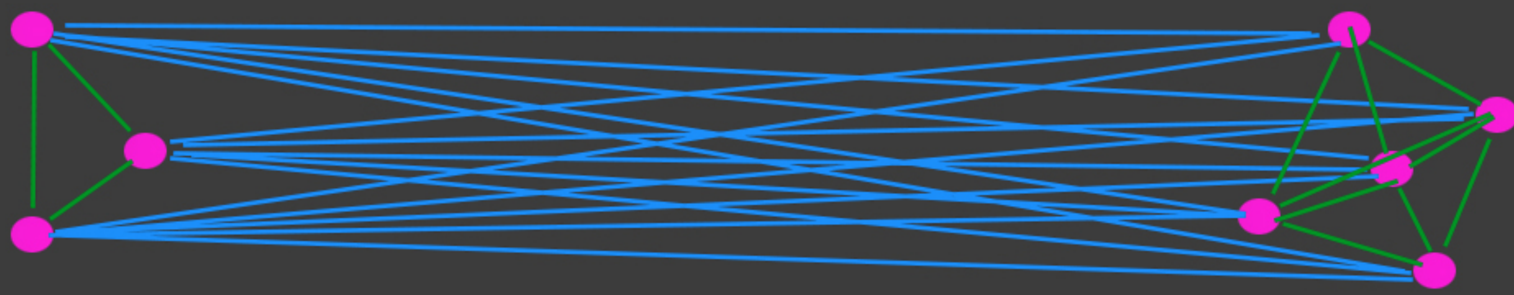
$$\|F\| = G\frac{mM}{r^2}$$

Johannes Kepler:

Orbital Period (Kepler's Third Law):

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Newton's Law of Gravity – Derived



m = 3

M = 5

Kepler's Third Law – Derived

Consider an object in a circular motion about the Sun.

Position (as a function of time t):

$$x(t) = r\cos(2\pi t/T) \qquad\qquad y(t) = r\sin(2\pi t/T)$$

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$$v_x(t) = -r\sin(2\pi t/T)\frac{2\pi}{T} \qquad v_y(t) = r\cos(2\pi t/T)\frac{2\pi}{T}$$

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Acceleration:

$$a_x(t) = -r\cos(2\pi t/T)\left(\frac{2\pi}{T}\right)^2 \qquad a_y(t) = -r\sin(2\pi t/T)\left(\frac{2\pi}{T}\right)^2$$

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Newton's laws:

$$\|a\| = r\frac{4\pi^2}{T^2} \qquad = \qquad \|F\|/m = \frac{GmM}{r^2 m} = \frac{GM}{r^2}$$

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Algebra:

$$T^2 = 4\pi^2\frac{r^3}{GM} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Kepler's Third Law – Simplified

Kepler's Law

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

If orbital radius r is measured in Sun/Earth distance (aka *astronomical units*) and orbital period is measured in years, then

$$1 = 2\pi\sqrt{\frac{1^3}{GM}}$$

Hence, Orbital Period (Kepler's Third Law):

$$T = \sqrt{r^3}$$

We can also write it like this:

$$r = T^{2/3}$$

So, if we know T, we also know r (in AU). How do we measure T?

How to Measure the Orbital Period

Suppose we look at a body (planet, asteroid, whatever) when it's exactly in the opposite direction from us as the Sun is.

Then, we watch it over and over night after night until it gets aligned again in that "opposite from the Sun" position.

Suppose that takes 2.5 years. So, in 2.5 years, that body went half way around the Sun.

Hence, it's period is 5 years.

Let's write 2.5 years as $n + t$ years, where $n = 2$ and $t = 0.5$.

With this notation, we can write

$$T = \frac{n+t}{t}.$$

That formula works in general.