Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to 180°.
Hence,

$$\alpha + \beta + \sigma = 180.$$

Also, because angle of incidence equals angle of reflection, we see that

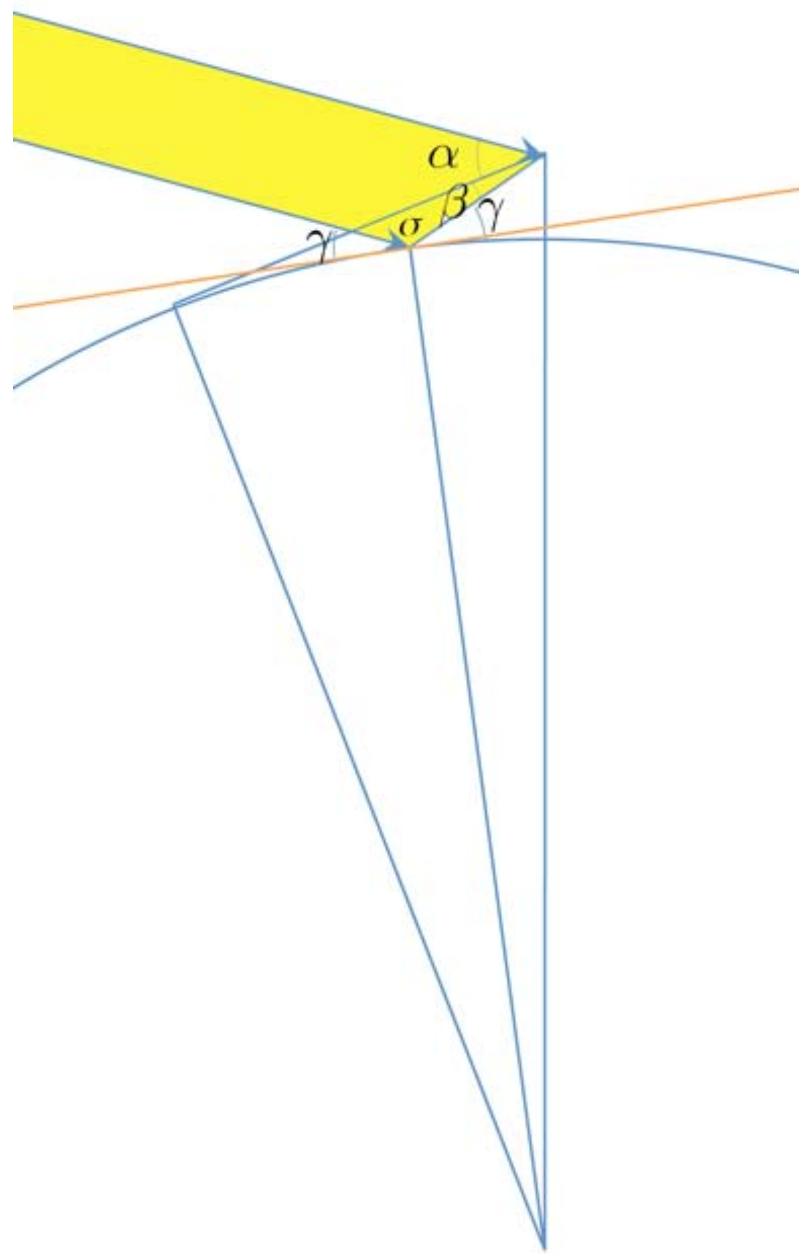
$$\gamma + \sigma + \gamma = 180.$$

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$

So, our second equation is

$$\alpha + \beta = 2\gamma.$$



Equation 3:

The distance from the center of the Earth to eye level is

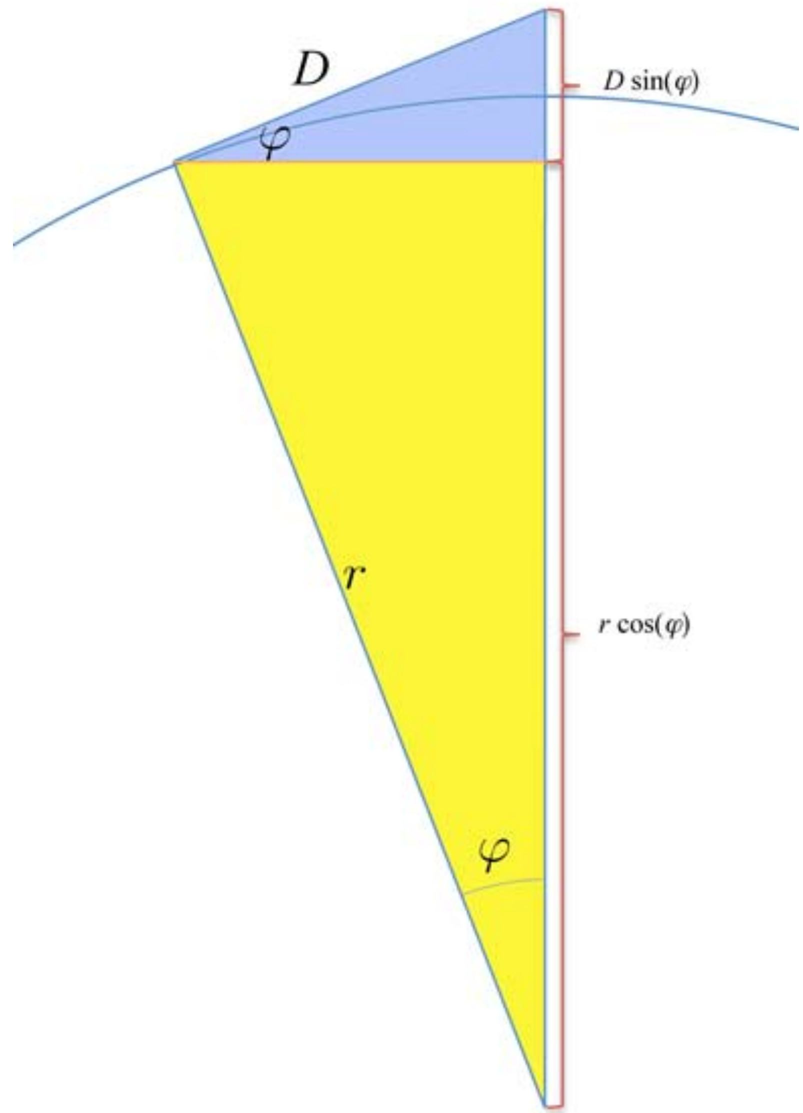
$$r + h.$$

But, it is also

$$D \sin(\varphi) + r \cos(\varphi).$$

Hence,

$$D \sin(\varphi) + r \cos(\varphi) = r + h.$$



Equation 4:

The “horizontal” distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

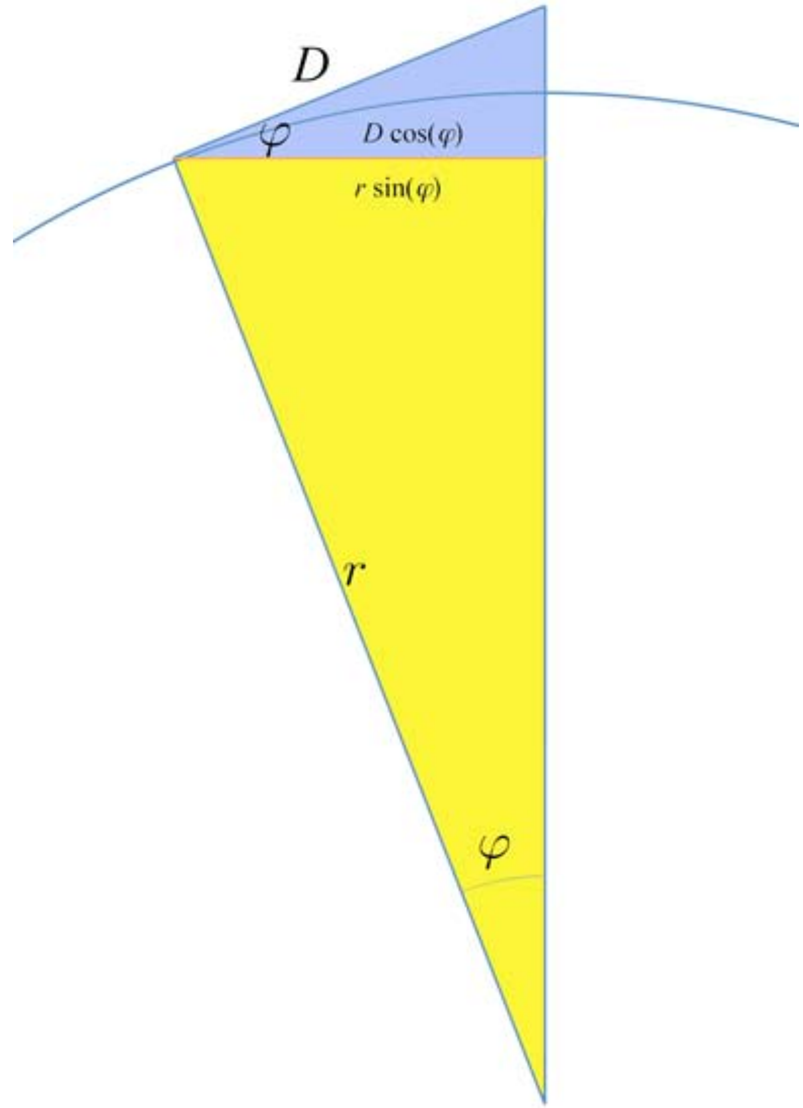
$$D\cos(\varphi)$$

and

$$r\sin(\varphi).$$

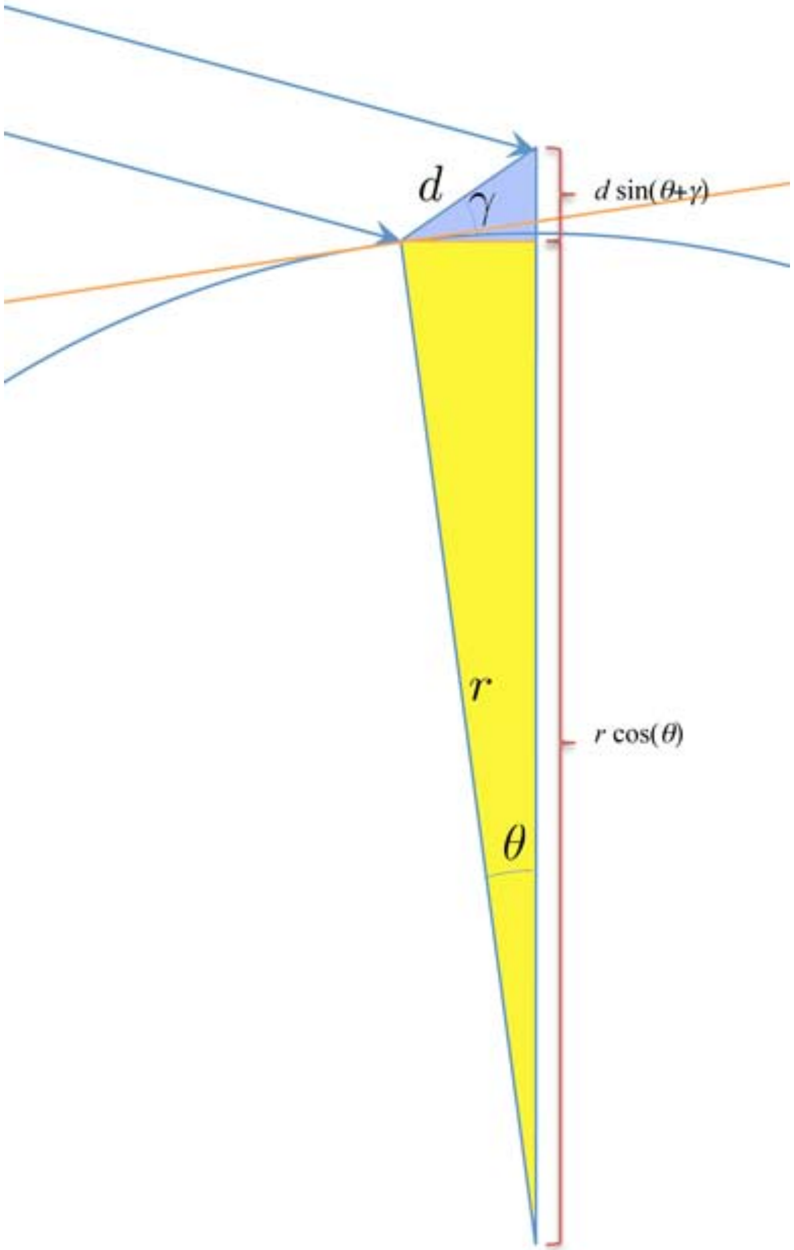
Hence,

$$D\cos(\varphi) = r\sin(\varphi).$$



Equation 5:

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h.$$

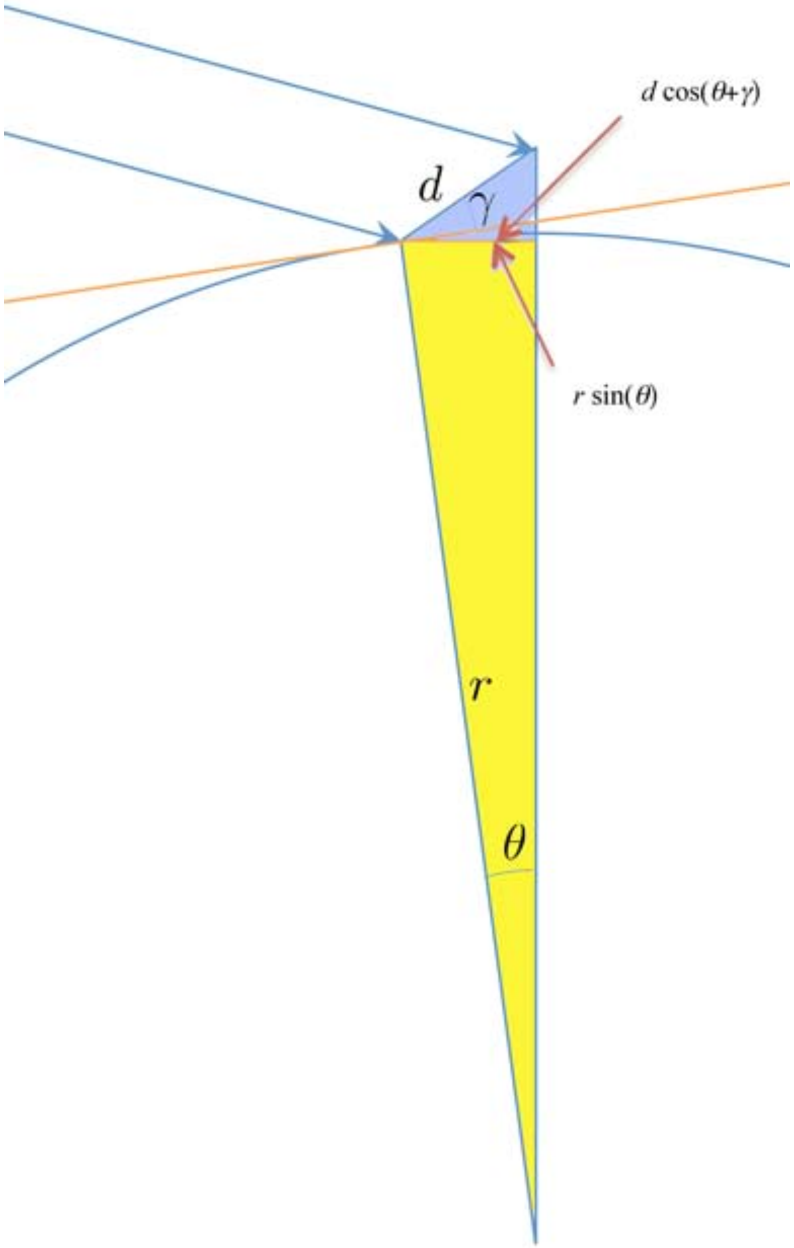


Equation 5 is analogous to Equation 3, using the "point of reflection" in place of the "horizon".

Equation 6:

Equation 6 is analogous to Equation 4 in the same way.

$$d\cos(\theta + \gamma) = r\sin(\theta).$$



Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \quad (1)$$

$$\alpha + \beta = 2\gamma \quad (2)$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \quad (3)$$

$$D \cos(\varphi) = r \sin(\varphi) \quad (4)$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \quad (5)$$

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Use (1) to change $\theta + \gamma$ to $\varphi + \beta$ in (5) and (6).

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And so on...

Three Equations Already "Solved":

$$\begin{aligned} \gamma &= (\alpha+\beta)/2 && (2) \\ D &= r\sin(\varphi)/\cos(\varphi) && (4) \\ d &= r\sin(\theta)/\cos(\varphi+\beta) && (6) \end{aligned}$$

Three Equations in Three Unknowns:

$$\begin{aligned} \varphi-\theta &= (\alpha-\beta)/2 && (1) \\ r &= (r+h)\cos(\varphi) && (3) \\ r\cos(\gamma) &= (r+h)\cos(\varphi+\beta) && (5) \end{aligned}$$

Divide (3) and (5) by $r+h$ and eliminate r:

$$\cos(\varphi) = \cos(\varphi+\beta)/\cos(\gamma)$$

Expand the cosine of the sum, replace $\sin(\varphi)$ with $\sqrt{1-\cos^2(\varphi)}$ and solve for $\cos(\varphi)$:

$$\cos(\varphi) = \frac{\sin(\beta)}{\sqrt{1-2\cos(\beta)\cos(\gamma)+\cos^2(\gamma)}}$$

Substitute this formula for $\cos(\varphi)$ into (3) and solve for r...

Answer for r (radius of Earth) is:

$$r = \frac{h}{\frac{\sqrt{1 - 2\cos\beta\cos\gamma + \cos^2\gamma}}{\sin\beta} - 1}$$

where

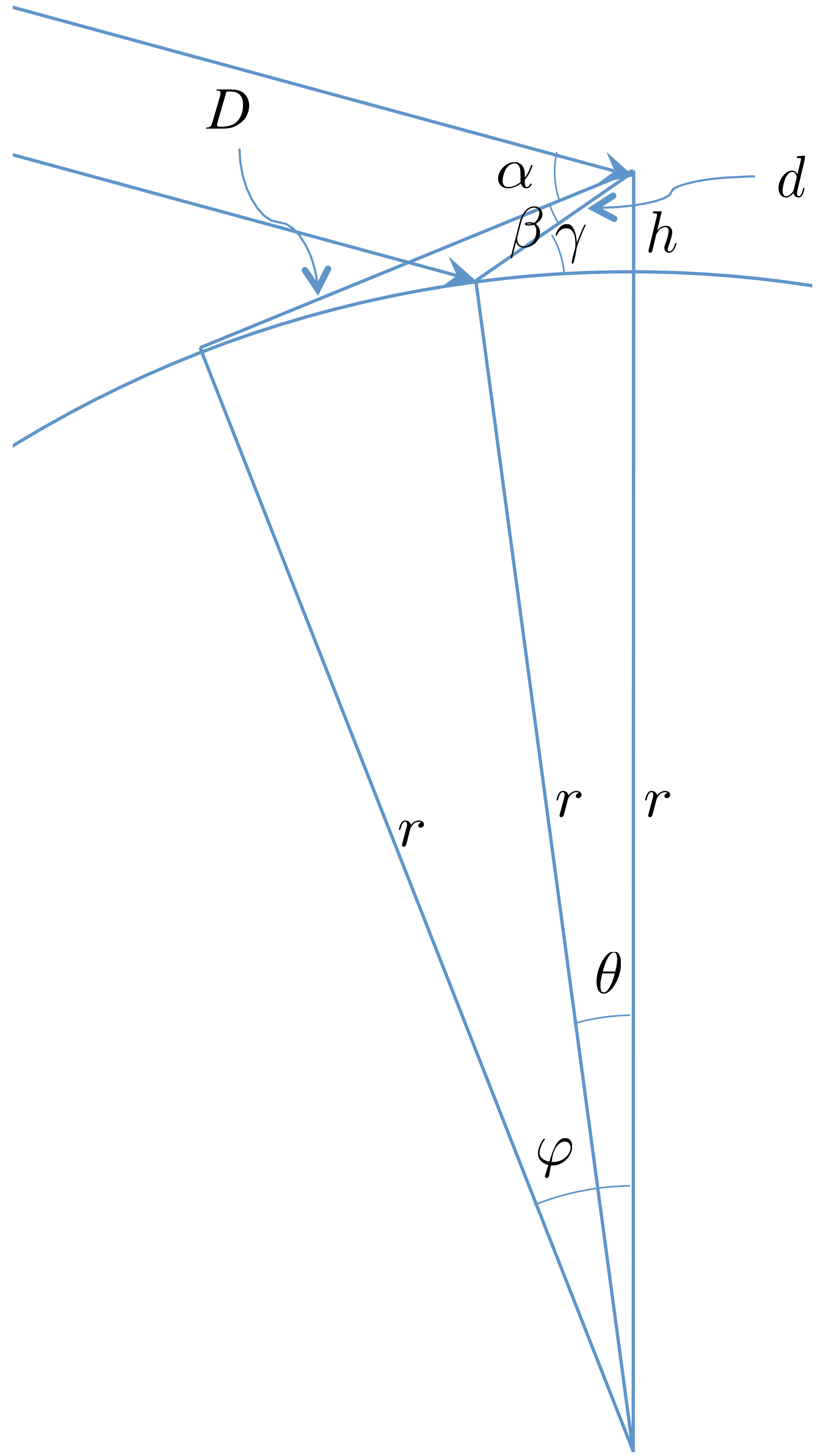
$$\gamma = \frac{\alpha + \beta}{2}.$$

Plugging in our values for α, β, and h, we get

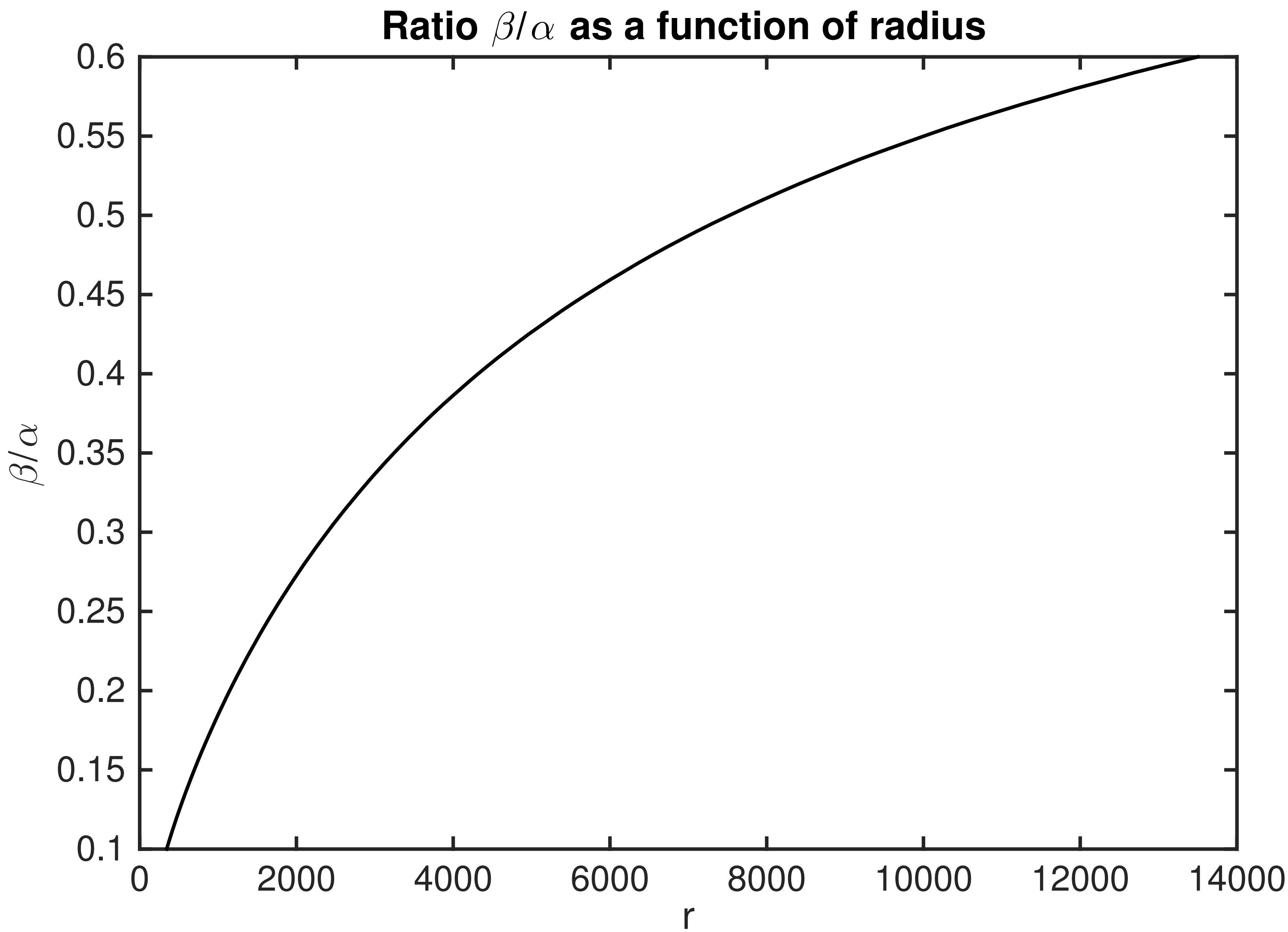
$$r = 4{,}977 \text{ miles.}$$

Recall that the right answer is 3,960 miles.
Fixing α and β, the height h that corresponds to this answer is:

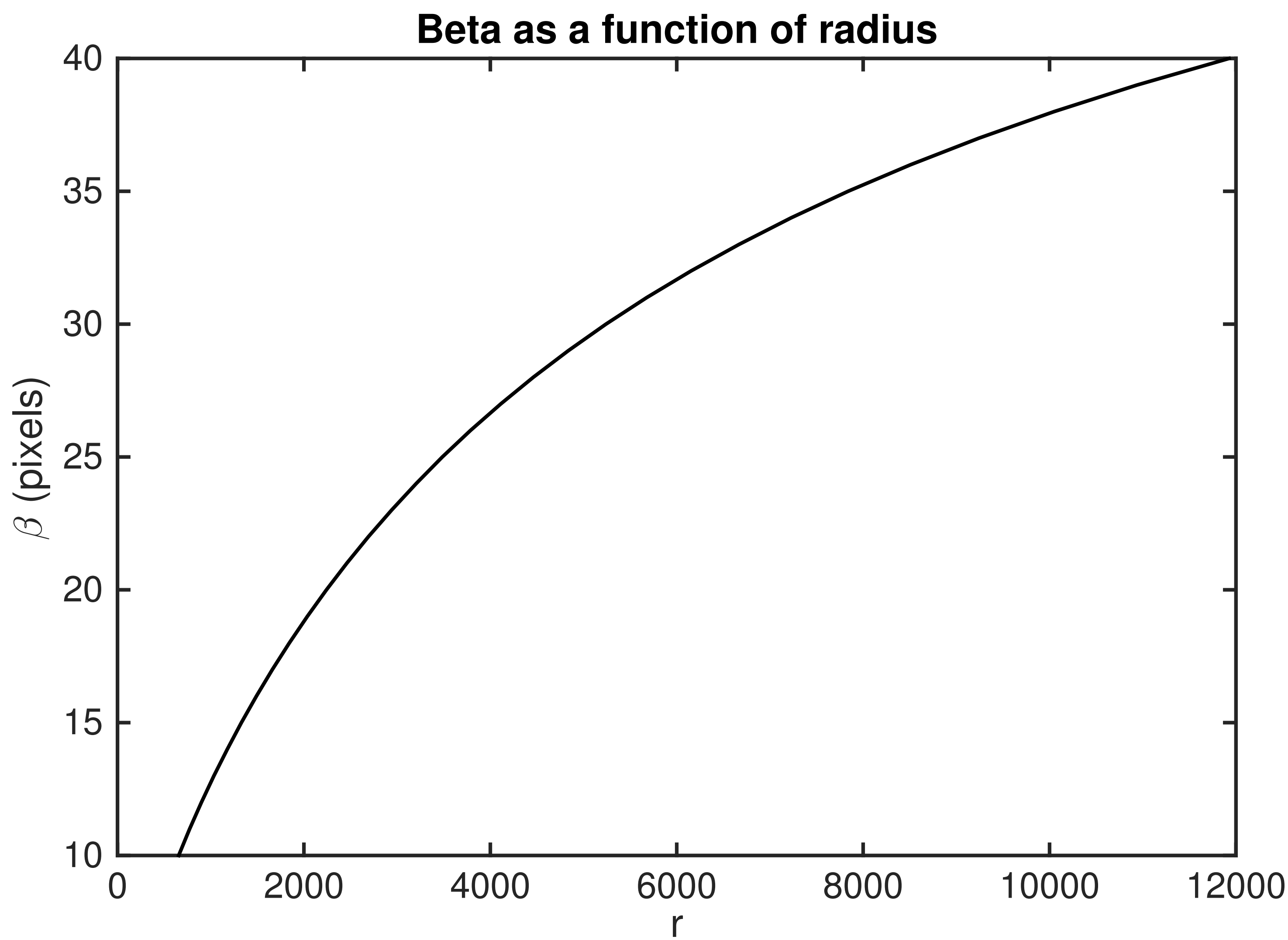
$$h = 7 \times \frac{3960}{4977} = 5.56 \text{ feet} = 5' \, 7''.$$



Fix α. How does the ratio β/α vary with r...



In terms of pixels...



Morals:

- Always be mindful of units.
- Always draw a picture and label things.
- If there are six unknowns, you need six (distinct) equations.
- A drawing need not be to scale; it can exagerate angles, distances, etc.

Conclusion: ALGEBRA AND GEOMETRY ARE BOTH FUN AND USEFUL.

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Using my iPhoneX, I took 16 pics to make a 360° panorama from the middle of the road in front of my house. I used Photoshop to assemble the pics:

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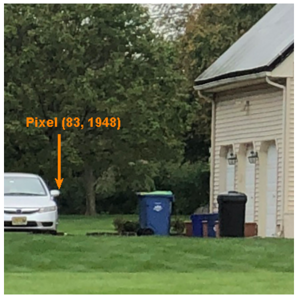

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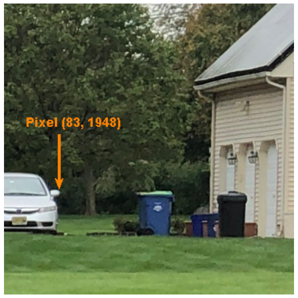

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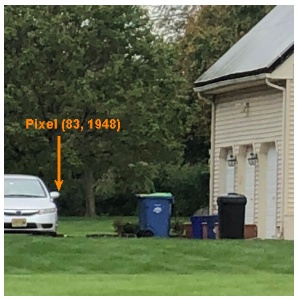

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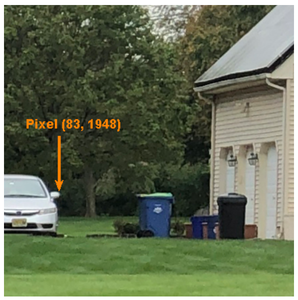

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Using these pixel measurements, we can compute the angular size of the Moon:

$$\text{Moon Size} = \frac{59}{38025} \times 360^\circ = 0.559^\circ$$

In arcminutes, this is

$$\begin{aligned}\text{Moon Size} &= 0.559^\circ \times 60 \, \frac{\text{arcmin}}{\text{deg}} \\ &= 33.5'\end{aligned}$$

Distance to the Moon.

From the partial lunar eclipse pics shown earlier, we see that the Earth is about 4 times bigger in diameter than the Moon.

And, using our sunset on Lake Michigan picture, we determined that the Earth is about 5000 miles in radius. So, we can estimate that the Moon's radius is about 1250 miles and its diameter is therefore about 2500 miles.

Let r denote the distance to the Moon, let θ denote the angular size of the Moon and let d denote the diameter of the Moon. We now have estimates of θ and d and so we can compute an estimate of r:

$$r = d/\sin(\theta) = 2500 \text{ miles}/\sin(0.559^\circ) = 257000 \text{ miles}$$

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PS. The Moon pic was taken in the evening on Sept. 19, 2020. According to Cartes du Ciel, the angular size of the Moon at that time was $33.26'$. If we consider that our pixel estimate could be off by $\pm 1/2$ pixels, we get the following range: $33.23'$ to $33.80'$. Bingo!