ORF 522: Lecture 7

Linear Programming: Chapter 7
Sensitivity and Parametric Analysis

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Restarting

Consider an optimal dictionary:

$$\begin{array}{rcl} \zeta &=& \zeta^* &-& {z^*_{\mathcal{N}}}^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &=& x^*_{\mathcal{B}} &-& B^{-1} N x_{\mathcal{N}}. \end{array}$$

Recall definitions of $x^*_{\mathcal{B}}$, $z^*_{\mathcal{N}}$, and ζ^*:

$$\begin{array}{rcl} x^*_{\mathcal{B}} &=& B^{-1} b \\ z^*_{\mathcal{N}} &=& (B^{-1} N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \\ \zeta^* &=& c^T_{\mathcal{B}} B^{-1} b. \end{array}$$

Now, suppose objective coefficients change from c to $\tilde{c}$.
To adjust current dictionary,

- recompute $z^*_{\mathcal{N}}$, and
- recompute ζ^*.

Note that $x^*_{\mathcal{B}}$ remains unchanged. Therefore,

- Adjusted dictionary is *primal feasible*.
- Apply primal simplex method.
- Likely to reach optimality quickly.

Had it been the right-hand sides b that changed, then

- Adjusted dictionary would be *dual feasible*.
- Could apply dual simplex method.

Ranging

Given an optimal dictionary:

$$\begin{array}{rcrcl} \zeta & = & \zeta^* & - & {z_{\mathcal{N}}^*}^T x_{\mathcal{N}} \\ x_{\mathcal{B}} & = & x_{\mathcal{B}}^* & - & B^{-1} N x_{\mathcal{N}}. \end{array}$$

Question: *If c were to change to $\tilde{c} = c + \mu \Delta c$, for what range of μ's does the current basis remain optimal?*

Recall that: $z_{\mathcal{N}}^* = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$

Therefore, dual variables change by $\mu \Delta z_{\mathcal{N}}$ where

$$\Delta z_{\mathcal{N}} = (B^{-1}N)^T \Delta c_{\mathcal{B}} - \Delta c_{\mathcal{N}}$$

We want: $z_{\mathcal{N}}^* + \mu \Delta z_{\mathcal{N}} \geq 0$

From familiar ratio tests, we get $\left(\min_{j \in \mathcal{N}} -\frac{\Delta z_j}{z_j^*} \right)^{-1} \leq \mu \leq \left(\max_{j \in \mathcal{N}} -\frac{\Delta z_j}{z_j^*} \right)^{-1}.$

Comments:

- A similar analysis works for changes to the right-hand side.
- An example is worked out in the text.

Ranging with the Pivot Tool

An initial dictionary:

obj	=	0.0			+	2.0	x1	+	1.0	x2
					+	-1.0	x1	+	-1.0	x2
w1	=	-1.0	+	1.0	-	-1.0	x1	-	-1.0	x2
w2	=	2.0	+	1.0	-	-1.0	x1	-	1.0	x2
w3	=	3.0	+	1.0	-	0.0	x1	-	1.0	x2
w4	=	5.0	+	1.0	-	1.0	x1	-	1.0	x2
w5	=	3.0	+	1.0	-	1.0	x1	-	0.0	x2
w6	=	2.0	+	1.0	-	1.0	x1	-	-1.0	x2

The optimal dictionary:

obj	=	8.0			+	-1.0	w4	+	-1.0	w5
					+	1.0	w4	+	0.0	w5
w1	=	4.0	+	2.0	-	1.0	w4	-	0.0	w5
w2	=	3.0	+	2.0	-	-1.0	w4	-	2.0	w5
w3	=	1.0	+	1.0	-	-1.0	w4	-	1.0	w5
w6	=	1.0	+	0.0	-	1.0	w4	-	-2.0	w5
x2	=	2.0	+	0.0	-	1.0	w4	-	-1.0	w5
x1	=	3.0	+	1.0	-	0.0	w4	-	1.0	w5

Question: *If the coefficient on x_2 in original problem were changed from 1 to $1 + \mu$ (and everything else remains unchanged), for what range of μ's does the current basis remain optimal?*

Ranging with the Pivot Tool–Continued

Set artificial rhs column to zeros.

Set artificial objective row to "x_2":

					–					
obj	=	8.0			+	-1.0	w4	+	-1.0	w5
					+	-1.0	w4	+	1.0	w5
w1	=	4.0	+	0.0	–	1.0	w4	–	0.0	w5
w2	=	3.0	+	0.0	–	-1.0	w4	–	2.0	w5
w3	=	1.0	+	0.0	–	-1.0	w4	–	1.0	w5
w6	=	1.0	+	0.0	–	1.0	w4	–	-2.0	w5
x2	=	2.0	+	0.0	–	1.0	w4	–	-1.0	w5
x1	=	3.0	+	0.0	–	0.0	w4	–	1.0	w5

-1.0 <= mu <= 1.0

The range of μ values is shown at the bottom of the pivot tool.

An Example

$$
\begin{array}{lrcrcrcr}
\text{maximize} & -3x_1 & + & 11x_2 & + & 2x_3 & & \\
\text{subj. to} & -x_1 & + & 3x_2 & & & \leq & 5 \\
& 3x_1 & + & 3x_2 & & & \leq & 4 \\
& & & 3x_2 & + & 2x_3 & \leq & 6 \\
& -3x_1 & & & - & 5x_3 & \leq & -4 \\
& & & & x_1, & x_2, \; x_3 & \geq & 0.
\end{array}
$$

Initial Dictionary:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & -3x_1 & + & 11x_2 & + & 2x_3 \\
\hline
w_1 & = & 5 & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & 3x_1 & & & + & 5x_3
\end{array}
$$

Note: neither primal nor dual feasible.

Perturb

Introduce a parameter μ and perturb:

$$
\begin{array}{rcrcrcrcrcrcr}
\zeta & = & & & & & -3x_1 & + & 11x_2 & + & 2x_3 \\
 & & & & & & -\mu x_1 & - & \mu x_2 & - & \mu x_3 \\
\hline
w_1 & = & 5 & + & \mu & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & + & \mu & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & + & \mu & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & \mu & + & 3x_1 & & & + & 5x_3
\end{array}
$$

For μ large, dictionary is **optimal**.

Question: For which μ values is dictionary optimal?

Answer:

$$
\begin{array}{rcrcrl}
5 & + & \mu & \geq & 0 & \\
4 & + & \mu & \geq & 0 & \\
6 & + & \mu & \geq & 0 & \\
-4 & + & \mu & \geq & 0 & *
\end{array}
\qquad
\begin{array}{rcrcrl}
-3 & - & \mu & \leq & 0 & \\
11 & - & \mu & \leq & 0 & * \\
2 & - & \mu & \leq & 0 & *
\end{array}
$$

Note: only those marked with (*) give inequalities that omit $\mu = 0$. Tightest:

$$\mu \geq 11$$

Achieved by: objective row perturbation on x_2. Let x_2 **enter.**

Who Leaves?

Do ratio test using current lowest μ value, i.e. $\mu = 11$:

$$\begin{array}{rcrcrcl} 5 & + & 11 & - & 3x_2 & \geq & 0 \\ 4 & + & 11 & - & 3x_2 & \geq & 0 \\ 6 & + & 11 & - & 3x_2 & \geq & 0 \\ -4 & + & 11 & & & \geq & 0 \end{array}$$

Tightest:

$$4 + 11 - 3x_2 \geq 0.$$

Achieved by: constraint containing basic variable w_2.

Let w_2 **leave.**

After the pivot:

$$\begin{array}{rclcrcrcrcr} \zeta & = & 14.67 & & & - & 14x_1 & - & 3.67w_2 & + & 2x_3 \\ & & & & & & & + & 0.33\mu w_2 & - & \mu x_3 \\ \hline w_1 & = & 1 & & & + & 4x_1 & + & w_2 & & \\ x_2 & = & 1.33 & + & 0.33\mu & - & x_1 & - & 0.33w_2 & & \\ w_3 & = & 2 & & & + & 3x_1 & + & w_2 & - & 2x_3 \\ w_4 & = & -4 & + & \mu & + & 3x_1 & & & + & 5x_3 \end{array}$$

Second Pivot

Using the *advanced* pivot tool, the current dictionary is:

obj	=	14.6667			+	-14.0	x1	+	-3.6667	w2	+	2.0	x3
					+	0.0	x1	+	0.3333	w2	+	-1.0	x3
w1	=	1.0	+	0.0	-	-4.0	x1	-	-1.0	w2	-	0.0	x3
x2	=	1.3333	+	0.3333	-	1.0	x1	-	0.3333	w2	-	0.0	x3
w3	=	2.0	+	0.0	-	-3.0	x1	-	-1.0	w2	-	2.0	x3
w4	=	-4.0	+	1.0	-	-3.0	x1	-	0.0	w2	-	-5.0	x3

Note: the parameter μ is not shown. **But it is there!** Question: For which μ values is dictionary optimal? Answer:

$$
\begin{array}{rcrcll}
-14 & & & \leq & 0 & \\
-3.67 & + & 0.33\mu & \leq & 0 & \\
2 & - & \mu & \leq & 0 & * \\
\hline
1 & & & \geq & 0 & \\
1.33 & + & 0.33\mu & \geq & 0 & \\
2 & & & \geq & 0 & \\
-4 & + & \mu & \geq & 0 & *
\end{array}
$$

Tightest lower bound:

$$\mu \geq 4$$

Achieved by: constraint containing basic variable w_4. Let w_4 **leave.**

Second Pivot–Continued

Who shall enter?

Recall the current dictionary:

obj	=	14.6667			+	-14.0	x1	+	-3.6667	w2	+	2.0	x3
					+	0.0	x1	+	0.3333	w2	+	-1.0	x3
w1	=	1.0	+	0.0	-	-4.0	x1	-	-1.0	w2	-	0.0	x3
x2	=	1.3333	+	0.3333	-	1.0	x1	-	0.3333	w2	-	0.0	x3
w3	=	2.0	+	0.0	-	-3.0	x1	-	-1.0	w2	-	2.0	x3
w4	=	-4.0	+	1.0	-	-3.0	x1	-	0.0	w2	-	-5.0	x3

Do *dual-type* ratio test using current lowest μ value, i.e. $\mu = 4$:

$$\begin{array}{rcrcrcl} 14 & + & 0 * 4 & - & 3y_4 & \geq & 0 \\ 3.67 & - & 0.33 * 4 & & & \geq & 0 \\ -2 & + & 1 * 4 & - & 5y_4 & \geq & 0 \end{array}$$

Tightest:

$$-2 + 1 * 4 - 5y_4 \geq 0.$$

Achieved by: objective term containing nonbasic variable x_3.

Let x_3 **enter.**

Third Pivot

The current dictionary is:

obj	=	16.2667			+	-15.2	x1	+	-3.6667	w2	+	0.4	w4
					+	0.6	x1	+	0.3333	w2	+	-0.2	w4
w1	=	1.0	+	0.0	-	-4.0	x1	-	-1.0	w2	-	0.0	w4
x2	=	1.3333	+	0.3333	-	1.0	x1	-	0.3333	w2	-	0.0	w4
w3	=	0.4	+	0.4	-	-4.2	x1	-	-1.0	w2	-	0.4	w4
x3	=	0.8	+	-0.2	-	0.6	x1	-	0.0	w2	-	-0.2	w4

Question: For which μ values is dictionary optimal? Answer:

$$
\begin{array}{rcrcll}
-15.2 & + & 0.6\mu & \leq & 0 & \\
-3.67 & + & 0.33\mu & \leq & 0 & \\
0.4 & - & 0.2\mu & \leq & 0 & * \\
\hline
1 & & & \geq & 0 & \\
1.33 & + & 0.33\mu & \geq & 0 & \\
0.4 & + & 0.4\mu & \geq & 0 & \\
0.8 & - & 0.2\mu & \geq & 0 &
\end{array}
$$

Tightest lower bound:

$$\mu \geq 2$$

Achieved by: objective term containing nonbasic variable w_4. Let w_4 **enter.**

Third Pivot–Continued

Who should leave?
Recall the current dictionary:

obj	=	16.2667			+	-15.2	x1	+	-3.6667	w2	+	0.4	w4
					+	0.6	x1	+	0.3333	w2	+	-0.2	w4
w1	=	1.0	+	0.0	−	-4.0	x1	−	-1.0	w2	−	0.0	w4
x2	=	1.3333	+	0.3333	−	1.0	x1	−	0.3333	w2	−	0.0	w4
w3	=	0.4	+	0.4	−	-4.2	x1	−	-1.0	w2	−	0.4	w4
x3	=	0.8	+	-0.2	−	0.6	x1	−	0.0	w2	−	-0.2	w4

Do *primal-type* ratio test using current lowest μ value, i.e. $\mu = 2$:

$$\begin{array}{rcrcrcl} 1 & + & 0 * 2 & & & \geq & 0 \\ 1.33 & + & 0.33 * 2 & & & \geq & 0 \\ 0.4 & + & 0.4 * 2 & - & 0.4 w_4 & \geq & 0 \\ 0.8 & - & 0.2 * 2 & + & 0.2 w_4 & \geq & 0 \end{array}$$

Tightest:

$$0.4 + 0.4 * 2 - 0.4 w_4 \geq 0$$

Achieved by: constraint containing basic variable w_3.
Variable w_3 should **leave.**

Fourth Pivot

The current dictionary is:

obj	=	16.6667			+	-11.0	x1	+	-2.6667	w2	+	-1.0	w3
					+	-1.5	x1	+	-0.1667	w2	+	0.5	w3
w1	=	1.0	+	0.0	−	-4.0	x1	−	-1.0	w2	−	0.0	w3
x2	=	1.3333	+	0.3333	−	1.0	x1	−	0.3333	w2	−	0.0	w3
w4	=	1.0	+	1.0	−	-10.5	x1	−	-2.5	w2	−	2.5	w3
x3	=	1.0	+	0.0	−	-1.5	x1	−	-0.5	w2	−	0.5	w3

It's **optimal**!
Also, the range of μ values includes $\mu = 0$:

$$\begin{array}{rcrcl} 1 & & & \geq & 0 \\ 1.33 & + & 0.33\mu & \geq & 0 \\ 1 & + & 1\mu & \geq & 0 \\ 1 & & & \geq & 0 \end{array} \qquad \begin{array}{rcrcl} -11 & - & 1.5\mu & \leq & 0 \\ -2.67 & - & 0.167\mu & \leq & 0 \\ -1 & + & 0.5\mu & \leq & 0 \end{array}$$

That is,

$$-1 \leq \mu \leq 2$$

Range of μ values is shown at bottom of pivot tool. Invalid ranges are highlighted in yellow.

Final Remarks

- The initial perturbation coefficients do not have to be all ones.
- In fact, for those objective-function/right-hand-side coefficients that are already of the correct sign, the perturbation can be *zero*.
- And, those that are positive can be chosen arbitrarily—even *randomly*.
- If all are chosen randomly with a distribution that has a density with respect to Lebesgue measure, then *degenerate pivots arise with probability zero*.
- Thought experiment:
  - μ starts at ∞.
  - In reducing μ, there are $n + m$ barriers.
  - At each iteration, one barrier is passed—the others move about randomly.
  - To get μ to zero, we must on average pass half the barriers.
  - Therefore, on average the algorithm should take $(m + n)/2$ iterations.