ORF 522: Lecture 22

Pricing American Options—Continued

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December 12, 2013

Slides last edited at 1:04pm on Thursday 12^{th} December, 2013

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Picking Up Where We Left Off

Primal Problem:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{j=0}^{\infty} v_j & \\ \text{subject to} & v_j \geq f_j & j \geq 0, \\ & v_j \geq \alpha \left(p v_{j+1} + q v_{j-1} \right) & j \geq 1. \end{array}$$

Associated Dual Problem:

$$\begin{array}{lllllll} \text{maximize} & \displaystyle\sum_{j=0}^{\infty} f_j y_j & & & & & \\ \text{subject to} & y_0 & & & -\alpha q z_1 & = 1 & \\ & y_1 & & +z_1 & -\alpha q z_2 & = 1 & \\ & y_j & -\alpha p z_{j-1} & +z_j & -\alpha q z_{j+1} & = 1 & j \geq 2, \\ & & & & y_j & \geq 0 & j \geq 0, \\ & & & & z_j & \geq 0 & j \geq 1. \end{array}$$

Main Result

Let v_j denote the optimal primal solution and y_j and z_j the optimal dual solution (i.e., we are dropping the usual "stars" that denote optimality).

Subject to some assumptions (given later), we have...

Theorem. There exists a j^* such that

$$\begin{aligned} v_0 &= f_0, & & \\ v_j &= \alpha(pv_{j+1} + qv_{j-1}) > f_j, & & \text{for } 0 < j < j^*, \\ v_j &= f_j > \alpha(pv_{j+1} + qv_{j-1}), & & \text{for } j^* \le j. \end{aligned}$$

Complementarity

Suppose there exists such a j^*.

Complementarity implies that

$$\begin{aligned} y_j &= 0, \qquad \text{for } 0 < j < j^*, && (1) \\ z_j &= 0, \qquad \text{for } j^* \leq j. && (2) \end{aligned}$$

Dual feasibility with (1) implies

$$\begin{aligned} z_1 - \alpha q z_2 &= 1, \\ -\alpha p z_{j-1} + z_j - \alpha q z_{j+1} &= 1, \quad 1 < j < j^*. \end{aligned}$$

Dual feasibility with (2) implies

$$\begin{aligned} y_{j^*} - \alpha p z_{j^*-1} &= 1, \\ y_j &= 1, \quad j > j^*. \end{aligned}$$

Second Order Difference Equations

The problem of solving the equalities has been reduced to a pair of second order difference equations with *Dirichlet boundary conditions*. The first difference equation is

$$\begin{aligned} v_j - \alpha(pv_{j+1} + qv_{j-1}) &= 0, \qquad 0 < j < j^*, & (3) \\ v_0 &= 0, & (4) \\ v_{j^*} &= f_{j^*} & (5) \end{aligned}$$

and the second one is

$$\begin{aligned} z_j - \alpha(pz_{j-1} + qz_{j+1}) &= 1, \qquad 0 < j < j^*, & (6) \\ z_0 &= 0, & (7) \\ z_{j^*} &= 0. & (8) \end{aligned}$$

Note that in (4) we used the fact that $f_0 = 0$ and in (6) we have added a new variable, z_0, which is just fixed to zero (by (7)). In this way we consolidate the difference equation for z_j to a more elegant form.

Explicit Solution for v_j

First, we solve the equation for v_j. To this end, suppose that

$$v_j = \xi^j$$

for some positive real number ξ. Substituting into the difference equation, we get

$$\xi^j - \alpha(p\xi^{j+1} + q\xi^{j-1}) = 0.$$

Dividing by ξ^{j-1}, we get a quadratic equation

$$-\alpha p\xi^2 + \xi - \alpha q = 0.$$

The two roots to this equation are

$$\xi_\pm = \frac{-1 \pm \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p}.$$

The general solution to the difference equation is therefore

$$v_j = c_+\xi_+^j + c_-\xi_-^j.$$

...continued...

From the first boundary condition (4), we get that $c_- = -c_+$.

This relation together with the second boundary condition (5) gives

$$c_+ = \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}}.$$

Hence,

$$v_j = f_{j^*} \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}, \qquad 0 < j < j^*. \tag{9}$$

Explicit Solution for z_j

We need a *particular solution* to the particular equation
and the *general solution* to the associated homogeneous equation.

For a particular solution, we try the simplest thing

$$z_j \equiv c.$$

Substituting into the difference equation, we discover that $c = 1/(1-\alpha)$.

The associated homogeneous equation is exactly the same as the equation for v_j except with p and q interchanged. Hence the general solution, which is the sum of the particular and the homogeneous, is given by

$$z_j = \frac{1}{1-\alpha} + c_+\zeta_+^j + c_-\zeta_-^j$$

where

$$\begin{aligned} \zeta_+ &= 1/\xi_- = \frac{-1+\sqrt{1-4\alpha^2 pq}}{-2\alpha q}, \\ \zeta_- &= 1/\xi_+ = \frac{-1-\sqrt{1-4\alpha^2 pq}}{-2\alpha q}. \end{aligned}$$

...continued...

Using the boundary conditions to eliminate the two undetermined constants, we get

$$z_j = \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}}\zeta_+^j - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}}\zeta_-^j\right) \Big/ (1 - \alpha), \qquad 0 < j < j^*.$$

To Summarize...

Assuming we know j^*, the solutions to the equalities are:

$$v_j = \begin{cases} 0 & j = 0 \\ f_{j^*}\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}, & 0 < j < j^* \\ f_j & j^* \leq j \end{cases}$$

$$z_j = \begin{cases} \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}}\zeta_+^j - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}}\zeta_-^j\right) \Big/ (1 - \alpha) & 0 < j < j^* \\ 0 & j^* \leq j \end{cases}$$

$$y_j = \begin{cases} 1 + \alpha q z_1 & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1} & j = j^* \\ 1 & j^* < j \end{cases}$$

Checking the Inequalities

We find j^* by checking that the inequalities hold:

$$y_j \geq 0 \qquad j \geq 0, \tag{10}$$

$$z_j \geq 0 \qquad j \geq 1, \tag{11}$$

$$v_j \geq f_j \qquad j \geq 0, \tag{12}$$

$$v_j \geq \alpha(pv_{j+1} + qv_{j-1}) \qquad j \geq 1. \tag{13}$$

Inequalities (11)

Inequalities (11): $\qquad z_j \geq 0 \qquad$ for all $j \geq 1$.

Follows trivially for $j \geq j^*$ from the formula given above for z_j.

To check them for $j < j^*$, we do a proof by contradiction.

So, suppose that $z_j < 0$ for some $0 < j < j^*$.

Then there exists a k at which z_k is negative and a local minimum:

$$z_k \leq z_{k-1} \quad \text{and} \quad z_k \leq z_{k+1}.$$

But, we also have

$$\begin{aligned} z_k &= 1 + \alpha(pz_{k-1} + qz_{k+1}) \\ &\geq 1 + \alpha(pz_k + qz_k) \\ &= 1 + \alpha z_k. \end{aligned}$$

Rearranging, we get $z_k \geq 1/(1-\alpha) > 0$, which contradicts the assumption that z_k is negative. Hence, inequalities (11) hold for all j.

(This is a simple example of a *minimum principle* as one encounters in harmonic analysis.)

Inequalities (10) and (13)

Inequalities (10): $$y_j \geq 0 \qquad \text{for all } j \geq 0.$$

These follow trivially from inequalities (11) and the formula for y_j.

Inequalities (13): $$v_j \geq \alpha(pv_{j+1} + qv_{j-1}) \qquad \text{for all } j \geq 1.$$

These hold trivially for $j < j^*$.

They also hold trivially for $j > j^*$ **provided we assume that** $\alpha p \leq 1/2$ and $\alpha q \leq 1/2$:

$$v_j = f_j = \frac{1}{2}(f_{j+1} + f_{j-1}) \geq \alpha p f_{j+1} + \alpha q f_{j-1} = \alpha\left(pv_{j+1} + qv_{j-1}\right).$$

We'll come back to the inequality (13) for $j = j^*$ after we consider inequalities (12).

Inequalities (12)

For $j \geq j^*$, these are trivial.

Furthermore, it follows immediately from (9) that $v_j \geq 0$ for all j.

Hence, we just need to check that $v_j \geq x_j - K$ for $j < j^*$.

In order to have these inequalities hold for $j < j^*$, **we need to pick**

$$j^* \in \mathcal{J} := \left\{ k : f_k \, \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} > f_{k-1} \right\}.$$

Of course, we need to assume that $\mathcal{J}$ is nonempty.

Clearly no k for which $x_k < K$ can belong to the set (because both f_k and f_{k-1} vanish).

For convenience, then, we assume that $K = j_K \Delta x$ for some j_K.

In that case, **we assume that $k = j_K + 1$ belongs to the set $\mathcal{J}$:**

$$f_{j_K+1} \, \frac{\xi_+^{j_K} - \xi_-^{j_K}}{\xi_+^{j_K+1} - \xi_-^{j_K+1}} > f_{j_K}.$$

...continued..

Let $h_j = x_j - K$.

With such a choice and the assumption that $j^* \in \mathcal{J}$, we have that $v_{j^*} = h_{j^*}$ and $v_{j^*-1} > h_{j^*-1}$.

Suppose that $v_{j'} < h_{j'}$ for some $j' < j^*$.

Then the sequence $u_j := v_j - h_j$ must have a local maximum at some point, say k, strictly between j' and j^*.

That is, $u_k \geq u_{k-1}$ and $u_k \geq u_{k+1}$. But, we also have

$$\begin{aligned} u_k \;=\; v_k - h_k \;&=\; \alpha(pv_{k+1} + qv_{k-1}) \;-\; \frac{1}{2}(h_{k+1} + h_{k-1}) \\ &\leq\; \frac{1}{2}(v_{k+1} + v_{k-1}) \;-\; \frac{1}{2}(h_{k+1} + h_{k-1}) \;=\; \frac{1}{2}(u_{k+1} + u_{k-1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq\; u_k. \end{aligned}$$

Hence, the inequalities must be equalities. Hence, $u_{k-1} = u_k = u_{k+1}$.

By induction, all the u_k's must be equal.

This contradicts the premise that $u_{j'} < 0$ and $u_{j^*-1} > 0$.

Hence, u_j can't have a local maximum and therefore v_j cannot dip below h_j.

Inequality (13) at $j = j^*$

Finally, to get inequality (13) for $j = j^*$, we need to assume that $j^* + 1 \notin \mathcal{J}$.

That is,

$$f_{j^*+1} \frac{\xi_+^{j^*} - \xi_-^{j^*}}{\xi_+^{j^*+1} - \xi_-^{j^*+1}} \leq f_{j^*}. \tag{14}$$

To see why, let w_j denote the solution to the difference equation

$$\begin{aligned} w_j - \alpha(p w_{j+1} + q w_{j-1}) &= 0, \qquad 0 < j, \\ w_0 &= 0, \\ w_{j^*} &= f_{j^*}. \end{aligned}$$

This is the same as (6)–(8) but extended to all j.

Clearly we have $v_{j^*} = w_{j^*}$ and $v_{j^*-1} = w_{j^*-1}$.

Hence, (13) at j^* will hold if and only if $v_{j^*+1} \leq w_{j^*+1}$:

$$f_{j^*+1} = v_{j^*+1} \leq w_{j^*+1} = f_{j^*} \frac{\xi_+^{j^*+1} - \xi_-^{j^*+1}}{\xi_+^{j^*} - \xi_-^{j^*}}$$

The resulting inequality is clearly equivalent to (14).

Summary

The theorem holds provided we make the extra assumptions that

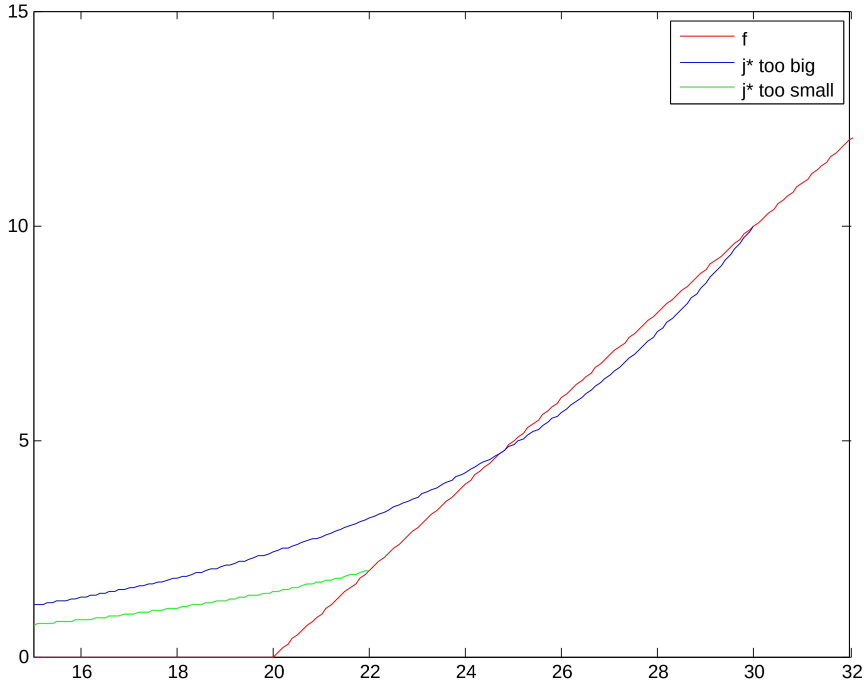
1. $\alpha p \leq 1/2$ and $\alpha q \leq 1/2$,
2. $K = j_K \Delta x$ for some integer j_K, and
3. $f_{j_K+1} \left(\xi_+^{j_K} - \xi_-^{j_K} \right) / \left(\xi_+^{j_K+1} - \xi_-^{j_K+1} \right) > f_{j_K}$.

With these assumptions, j^* must be chosen as

$$j^* := \max \left\{ k : f_k \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} > f_{k-1} \right\}.$$

Details can be found at:

`http://orfe.princeton.edu/~rvdb/tex/AmericanCallOption/perprev.pdf`



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Happy Holidays!