



# Linear Programming: Chapter 11

## Game Theory

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# Rock-Paper-Scissors

A two person game.

## *Rules.*

At the count of three declare one of:

Rock      Paper      Scissors

*Winner Selection.* Identical selection is a draw. Otherwise:

- Rock beats Scissors
- Paper beats Rock
- Scissors beats Paper

Check out Sam Kass' version: [Rock, Paper, Scissors, Lizard, Spock](#)

It was featured recently on [The Big Bang Theory](#).

# Payoff Matrix

Payoffs are *from* row player *to* column player:

$$A = \begin{array}{c} R \\ P \\ S \end{array} \begin{array}{ccc} R & P & S \\ \left[ \begin{array}{ccc} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array} \right] \end{array}$$

*Note:* Any *deterministic* strategy employed by either player can be defeated systematically by the other player.

# Two-Person Zero-Sum Games

*Given:*  $m \times n$  matrix  $A$ .

- *Row player* (rowboy) selects a *strategy*  $i \in \{1, \dots, m\}$ .
- *Col player* (colgirl) selects a *strategy*  $j \in \{1, \dots, n\}$ .
- Rowboy pays colgirl  $a_{ij}$  dollars.

*Note:* The rows of  $A$  represent deterministic strategies for rowboy, while columns of  $A$  represent deterministic strategies for colgirl.

**Deterministic strategies can be bad.**

# Randomized Strategies.

- Suppose rowboy picks  $i$  with probability  $y_i$ .
- Suppose colgirl picks  $j$  with probability  $x_j$ .

Throughout,  $x = [x_1 \ x_2 \ \cdots \ x_n]^T$  and  $y = [y_1 \ y_2 \ \cdots \ y_m]^T$  will denote *stochastic vectors*:

$$\begin{aligned}x_j &\geq 0, & j = 1, 2, \dots, n \\ \sum_j x_j &= 1.\end{aligned}$$

If rowboy uses random strategy  $y$  and colgirl uses  $x$ , then *expected payoff* from rowboy to colgirl is

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$

# Colgirl's Analysis

Suppose colgirl were to adopt strategy  $x$ .

Then, rowboy's best defense is to use  $y$  that minimizes  $y^T Ax$ :

$$\min_y y^T Ax$$

And so colgirl should choose that  $x$  which maximizes these possibilities:

$$\max_x \min_y y^T Ax$$

# Solving Max-Min Problems as LPs

Inner optimization is easy:

$$\min_y y^T Ax = \min_i e_i^T Ax$$

( $e_i$  denotes the vector that's all zeros except for a one in the  $i$ -th position—that is, deterministic strategy  $i$ ).

*Note:* Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

$$\begin{aligned} \max \quad & (\min_i e_i^T Ax) \\ \sum_j x_j &= 1, \\ x_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

# Reduction to a Linear Programming Problem

Introduce a scalar variable  $v$  representing the value of the inner minimization:

$$\begin{aligned} \max v \\ v &\leq e_i^T Ax, \quad i = 1, 2, \dots, m, \\ \sum_j x_j &= 1, \\ x_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

Writing in pure matrix-vector notation:

$$\begin{aligned} \max v \\ ve - Ax &\leq 0 \\ e^T x &= 1 \\ x &\geq 0 \end{aligned}$$

( $e$  denotes the vector of all ones).

# Finally, in Block Matrix Form

$$\max \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x \geq 0$$

$v$  free

# Rowboy's Perspective

Similarly, rowboy seeks  $y^*$  attaining:

$$\min_y \max_x y^T A x$$

which is equivalent to:

$$\begin{aligned} \min u \\ u e - A^T y &\geq 0 \\ e^T y &= 1 \\ y &\geq 0 \end{aligned}$$

# Rowboy's Problem in Block-Matrix Form

$$\min \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix}$$

$$\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \begin{matrix} \geq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y \geq 0$$

$u$  free

*Note:* Rowboy's problem is dual to colgirl's.

# MiniMax Theorem

Let  $x^*$  denote colgirl's solution to her max–min problem.

Let  $y^*$  denote rowboy's solution to his min–max problem.

Then

$$\max_x y^{*T} Ax = \min_y y^T Ax^*.$$

Proof.

From *Strong Duality Theorem*, we have

$$u^* = v^*.$$

Also,

$$v^* = \min_i e_i^T Ax^* = \min_y y^T Ax^*$$

$$u^* = \max_j y^{*T} Ae_j = \max_x y^{*T} Ax$$

QED

# AMPL Model

```
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
```

# AMPL Data

```
data;
set ROWS := P S R;
set COLS := P S R;
param A: P S R:=
    P 0 1 -2
    S -3 0 4
    R 5 -6 0
    ;

solve;
printf {j in COLS}: "    %3s %10.7f \n", j, 102*x[j];
printf {i in ROWS}: "    %3s %10.7f \n", i, 102*ineqs[i];
printf: "Value = %10.7f \n", 102*v;
```

# AMPL Output

```
AMPL gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective -0.1568627451
  dual objective -0.1568627451
    P 40.0000000
    S 36.0000000
    R 26.0000000
    P 62.0000000
    S 27.0000000
    R 13.0000000
Value = -16.0000000
```

# Dual of Problems in General Form

Consider:

$$\begin{aligned} \max c^T x \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Rewrite equality constraints as pairs of inequalities:

$$\begin{aligned} \max c^T x \\ Ax &\leq b \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Put into block-matrix form:

$$\begin{aligned} \max c^T x \\ \begin{bmatrix} A \\ -A \end{bmatrix} x &\leq \begin{bmatrix} b \\ -b \end{bmatrix} \\ x &\geq 0 \end{aligned}$$

Dual is:

$$\begin{aligned} \min \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \\ \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} &\geq c \\ y^+, y^- &\geq 0 \end{aligned}$$

Which is equivalent to:

$$\begin{aligned} \min b^T (y^+ - y^-) \\ A^T (y^+ - y^-) &\geq c \\ y^+, y^- &\geq 0 \end{aligned}$$

Finally, letting  $y = y^+ - y^-$ , we get

$$\begin{aligned} \min b^T y \\ A^T y &\geq c \\ y &\text{ free.} \end{aligned}$$

# Moral

- Equality constraints  $\implies$  free variables in dual.
- Inequality constraints  $\implies$  nonnegative variables in dual.

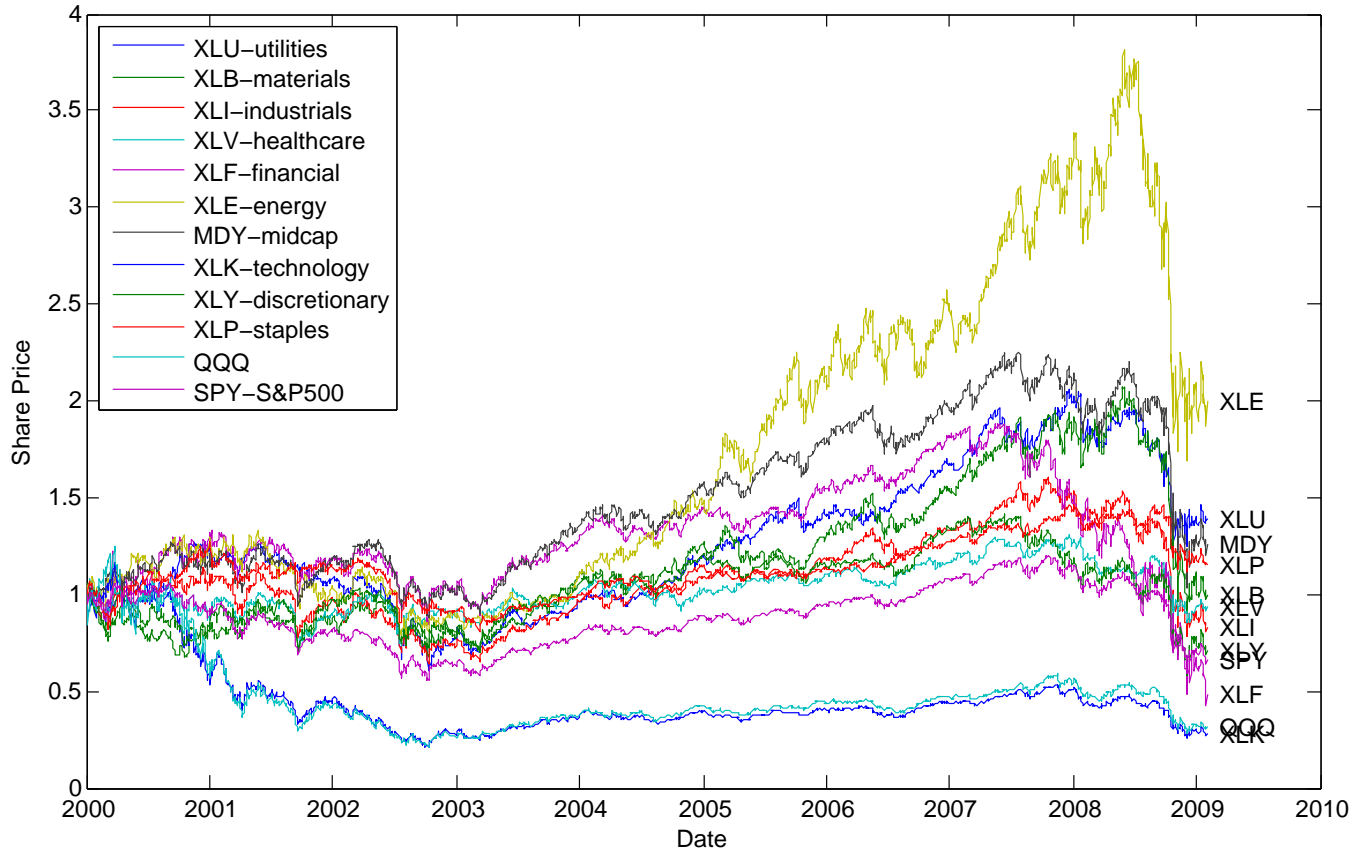
## Corollary:

- Free variables  $\implies$  equality constraints in dual.
- Nonnegative variables  $\implies$  inequality constraints in dual.

# A Real-World Example

## The Ultra-Conservative Investor

Consider again the historical investment data ( $S_j(t)$ ):



We can let  $R_{j,t} = S_j(t)/S_j(t-1)$  and view  $R$  as a payoff matrix in a game between *Fate* and the *Investor*.

# Fate's Conspiracy

The columns represent pure strategies for our conservative investor.

The rows represent how history might repeat itself.

Of course, for tomorrow, Fate won't just repeat a previous year but, rather, will present some mixture of these previous years.

Likewise, the investor won't put all of her money into one asset. Instead she will put a certain fraction into each.

Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

## Investor's Optimal Asset Mix:

XLP	90.7
QQQQ	9.3

## Mean, old Fate's Mix:

2008-10-08	37.6
2008-11-28	62.4

The value of the game is the investor's expected return, 94.3%, which is actually a loss of 5.7%.