

Linear Programming: Chapter 6

Matrix Notation

Robert J. Vanderbei

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Operations Research and Financial Engineering
Princeton University
Princeton, NJ 08544
<http://www.princeton.edu/~rvdb>

An Example

Consider

$$\begin{array}{ll} \text{maximize} & 3x_1 + 4x_2 - 2x_3 \\ \text{subject to} & x_1 + 0.5x_2 - 5x_3 \leq 2 \\ & 2x_1 - x_2 + 3x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Add slacks (using x 's for slack variables):

$$\begin{array}{rcl} x_1 + 0.5x_2 - 5x_3 + x_4 & = & 2 \\ 2x_1 - x_2 + 3x_3 + x_5 & = & 3. \end{array}$$

Cast constraints into matrix notation:

$$\begin{bmatrix} 1 & 0.5 & -5 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Similarly cast objective function:

$$\begin{bmatrix} 3 \\ 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} .$$

In general, we have:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

Down the Road

Basic Variables: x_2, x_5 .

Nonbasic Variables: x_1, x_3, x_4 .

$$\begin{aligned} Ax &= \begin{bmatrix} x_1 + 0.5x_2 - 5x_3 + x_4 \\ 2x_1 - x_2 + 3x_3 + x_5 \end{bmatrix} \\ &= \begin{bmatrix} 0.5x_2 + x_1 - 5x_3 + x_4 \\ -x_2 + x_5 + 2x_1 + 3x_3 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \\ &= Bx_B + Nx_N. \end{aligned}$$

General Matrix Notation

Up to a rearrangement of columns,

$$A \stackrel{\text{R}}{=} [B \quad N]$$

Similarly, rearrange rows of x and c :

$$x \stackrel{\text{R}}{=} \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix} \quad c \stackrel{\text{R}}{=} \begin{bmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{bmatrix}$$

Constraints:

$$Ax = b \quad \iff \quad Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b$$

Objective:

$$\zeta = c^T x \quad \iff \quad c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}}$$

Matrix B is $m \times m$ and **invertible**! Why?

Express $x_{\mathcal{B}}$ and ζ in terms of $x_{\mathcal{N}}$:

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Nx_{\mathcal{N}}$$

$$\begin{aligned}\zeta &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1}b - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}}.\end{aligned}$$

Dictionary in Matrix Notation

$$\begin{aligned}\zeta &= c_{\mathcal{B}}^T B^{-1}b - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= B^{-1}b - B^{-1}Nx_{\mathcal{N}}.\end{aligned}$$

Example Revisited

$$B = \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \implies B^{-1} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -10 & 2 \\ 4 & -7 & 2 \end{bmatrix}$$

$$(B^{-1}N)^T c_B - c_N = \begin{bmatrix} 2 & 4 \\ -10 & -7 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -38 \\ 8 \end{bmatrix}$$

$$c_B^T B^{-1}b = [4 \ 0] \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 16$$

Sanity Check

$$\begin{array}{r} \zeta = \quad 3x_1 + 4x_2 - 2x_3 \\ \hline x_4 = 2 - x_1 - 0.5x_2 + 5x_3 \\ x_5 = 3 - 2x_1 + x_2 - 3x_3. \end{array}$$

Let x_2 enter and x_4 leave.

$$\begin{array}{r} \zeta = 16 - 5x_1 - 8x_4 + 38x_3 \\ \hline x_2 = 4 - 2x_1 - 2x_4 + 10x_3 \\ x_5 = 7 - 4x_1 - 2x_4 + 7x_3. \end{array}$$

Dual Stuff

Associated Primal Solution:

$$\begin{aligned}x_{\mathcal{N}}^* &= 0 \\x_{\mathcal{B}}^* &= B^{-1}b\end{aligned}$$

Dual Variables:

$$\begin{aligned}(x_1, \dots, x_n, w_1, \dots, w_m) &\longrightarrow (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \\(z_1, \dots, z_n, y_1, \dots, y_m) &\longrightarrow (z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m})\end{aligned}$$

Associated Dual Solution:

$$\begin{aligned}z_{\mathcal{B}}^* &= 0 \\z_{\mathcal{N}}^* &= (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\end{aligned}$$

Associated Solution Value:

$$\zeta^* = c_{\mathcal{B}}^T B^{-1}b$$

Primal Dictionary:

$$\begin{aligned}\zeta &= \zeta^* - z_{\mathcal{N}}^{*T} x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}}.\end{aligned}$$

Dual Dictionary:

$$\begin{aligned}-\xi &= -\zeta^* - x_{\mathcal{B}}^{*T} z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z_{\mathcal{N}}^* + (B^{-1} N)^T z_{\mathcal{B}}.\end{aligned}$$

What have we gained?

1. A notation for doing proofs—no more proof by example.
2. Serious implementations of the simplex method avoid ever explicitly forming $B^{-1}N$.
Reason:

- The matrices B and N are sparse.
- But B^{-1} is likely to be fully dense.
- Even if B^{-1} is not dense, $B^{-1}N$ is going to be worse.
- It's better simply to solve

$$Bx_{\mathcal{B}} = b - Nx_{\mathcal{N}}$$

efficiently.

- This is subject of next chapter.
- We'll skip it this year.

Primal Simplex

Suppose $x_{\mathcal{B}}^* \geq 0$

while $(z_{\mathcal{N}}^* \not\geq 0)$ {

pick $j \in \{j \in \mathcal{N} : z_j^* < 0\}$

$$\Delta x_{\mathcal{B}} = B^{-1} N e_j$$

$$t = \left(\max_{i \in \mathcal{B}} \frac{\Delta x_i}{x_i^*} \right)^{-1}$$

pick $i \in \operatorname{argmax}_{i \in \mathcal{B}} \frac{\Delta x_i}{x_i^*}$

$$\Delta z_{\mathcal{N}} = -(B^{-1} N)^T e_i$$

$$s = \frac{z_j^*}{\Delta z_j}$$

$$x_j^* \leftarrow t, \quad x_{\mathcal{B}}^* \leftarrow x_{\mathcal{B}}^* - t \Delta x_{\mathcal{B}}$$

$$z_i^* \leftarrow s, \quad z_{\mathcal{N}}^* \leftarrow z_{\mathcal{N}}^* - s \Delta z_{\mathcal{N}}$$

$$\mathcal{B} \leftarrow \mathcal{B} \setminus \{i\} \cup \{j\}$$

}

Dual Simplex

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}

Symmetry Lost

B is $m \times m$. Why not $n \times n$? What's go'in on?

A Problem and Its Dual

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0 \end{array}$$

Add Slacks

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax + w = b \\ & x, w \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y - z = c \\ & y, z \geq 0 \end{array}$$

New Notations for Primal

$$\bar{A} = [A \ I], \quad \bar{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x \\ w \end{bmatrix}$$

New Notations for Dual

$$\hat{A} = \begin{bmatrix} -I & A^T \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} z \\ y \end{bmatrix}$$

Primal and Dual

$$\begin{array}{ll} \text{maximize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A} \bar{x} = b \\ & \bar{x} \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & \hat{b}^T \hat{z} \\ \text{subject to} & \hat{A} \hat{z} = c \\ & \hat{z} \geq 0 \end{array}$$

Symmetry Regained...

On the Primal Side:

$$[A \ I] \stackrel{R}{=} [\bar{N} \ \bar{B}]$$

On the Dual Side:

$$[-I \ A^T] \stackrel{R}{=} [\hat{B} \ \hat{N}]$$

Now Multiply:

$$\begin{aligned} \bar{A}\hat{A}^T &= [\bar{N} \ \bar{B}] \begin{bmatrix} \hat{B}^T \\ \hat{N}^T \end{bmatrix} \\ &= \bar{N}\hat{B}^T + \bar{B}\hat{N}^T \end{aligned}$$

And Again:

$$\begin{aligned} \bar{A}\hat{A}^T &= [A \ I] \begin{bmatrix} -I \\ A \end{bmatrix} \\ &= -A + A = 0 \end{aligned}$$

The Two Expressions Must Be Equal:

$$\bar{N}\hat{B}^T + \bar{B}\hat{N}^T = 0$$

But That's the Negative Transpose Property:

$$\bar{B}^{-1}\bar{N} = -\left(\hat{B}^{-1}\hat{N}\right)^T$$