

Linear Programming: Chapter 2

The Simplex Method

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Rewrite with slack variables

$$\begin{aligned} \text{maximize} \quad & \zeta = -x_1 + 3x_2 - 3x_3 \\ \text{subject to} \quad & w_1 = 7 - 3x_1 + x_2 + 2x_3 \\ & w_2 = 3 + 2x_1 + 4x_2 - 4x_3 \\ & w_3 = 4 - x_1 + 2x_3 \\ & w_4 = 8 + 2x_1 - 2x_2 - x_3 \\ & w_5 = 5 - 3x_1 \\ & x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0. \end{aligned}$$

Notes:

- This *layout* is called a *dictionary*.
- Setting x_1 , x_2 , and x_3 to 0, we can read off the values for the other variables: $w_1 = 7$, $w_2 = 3$, etc. This specific solution is called a *dictionary solution*.
- Dependent variables, on the left, are called *basic variables*.
- Independent variables, on the right, are called *nonbasic variables*.

Dictionary Solution is Feasible

$$\begin{aligned} \text{maximize} \quad & \zeta = -x_1 + 3x_2 - 3x_3 \\ \text{subject to} \quad & w_1 = 7 - 3x_1 + x_2 + 2x_3 \\ & w_2 = 3 + 2x_1 + 4x_2 - 4x_3 \\ & w_3 = 4 - x_1 + 2x_3 \\ & w_4 = 8 + 2x_1 - 2x_2 - x_3 \\ & w_5 = 5 - 3x_1 \\ & x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0. \end{aligned}$$

Notes:

- All the variables in the current dictionary solution are nonnegative.
- Such a solution is called *feasible*.
- The initial dictionary solution need not be feasible—we were just lucky above.

Simplex Method—First Iteration

		Current Dictionary								
obj =	0.0	+	-1.0	x1	+	3.0	x2	+	-3.0	x3
w1 =	7.0	-	3.0	x1	-	-1.0	x2	-	-2.0	x3
w2 =	3.0	-	-2.0	x1	-	-4.0	x2	-	4.0	x3
w3 =	4.0	-	1.0	x1	-	0.0	x2	-	-2.0	x3
w4 =	8.0	-	-2.0	x1	-	2.0	x2	-	1.0	x3
w5 =	5.0	-	3.0	x1	-	0.0	x2	-	0.0	x3

- If x_2 increases, obj goes *up*.
- How much can x_2 increase? Until w_4 decreases to zero.
- Do it. End result: $x_2 > 0$ whereas $w_4 = 0$.
- That is, x_2 must become *basic* and w_4 must become *nonbasic*.
- Algebraically rearrange equations to, in the words of Jean-Luc Picard, "Make it so."
- This is a *pivot*.

A Pivot: $x_2 \leftrightarrow w_4$

		Current Dictionary								
obj =	0.0	+	-1.0	x1	+	3.0	x2	+	-3.0	x3
w1 =	7.0	-	3.0	x1	-	-1.0	x2	-	-2.0	x3
w2 =	3.0	-	-2.0	x1	-	-4.0	x2	-	4.0	x3
w3 =	4.0	-	1.0	x1	-	0.0	x2	-	-2.0	x3
w4 =	8.0	-	-2.0	x1	-	2.0	x2	-	1.0	x3
w5 =	5.0	-	3.0	x1	-	0.0	x2	-	0.0	x3

becomes

		Current Dictionary								
obj =	12.0	+	2.0	x1	+	-1.5	w4	+	-4.5	x3
w1 =	11.0	-	2.0	x1	-	0.5	w4	-	-1.5	x3
w2 =	19.0	-	-6.0	x1	-	2.0	w4	-	6.0	x3
w3 =	4.0	-	1.0	x1	-	0.0	w4	-	-2.0	x3
x2 =	4.0	-	-1.0	x1	-	0.5	w4	-	0.5	x3
w5 =	5.0	-	3.0	x1	-	0.0	w4	-	0.0	x3

Simplex Method—Second Pivot

Here's the dictionary after the first pivot:

		Current Dictionary								
obj =	12.0	+	2.0	x1	+	-1.5	w4	+	-4.5	x3
w1 =	11.0	-	2.0	x1	-	0.5	w4	-	-1.5	x3
w2 =	19.0	-	-6.0	x1	-	2.0	w4	-	6.0	x3
w3 =	4.0	-	1.0	x1	-	0.0	w4	-	-2.0	x3
x2 =	4.0	-	-1.0	x1	-	0.5	w4	-	0.5	x3
w5 =	5.0	-	3.0	x1	-	0.0	w4	-	0.0	x3

- Now, let x_1 increase.
- Of the basic variables, w_5 hits zero first.
- So, x_1 enters and w_5 leaves the basis.
- New dictionary is...

Simplex Method—Final Dictionary

Current Dictionary							
obj =	$46/3$	+	$-2/3$	w5 +	$-3/2$	w4 +	$-9/2$ x3
w1 =	$23/3$	-	$-2/3$	w5 -	$1/2$	w4 -	$-3/2$ x3
w2 =	29	-	2	w5 -	2	w4 -	6 x3
w3 =	$7/3$	-	$-1/3$	w5 -	0	w4 -	-2 x3
x2 =	$17/3$	-	$1/3$	w5 -	$1/2$	w4 -	$1/2$ x3
x1 =	$5/3$	-	$1/3$	w5 -	0	w4 -	0 x3

- It's optimal (no pink)!
- Click [here](#) to practice the simplex method.
- For instructions, click [here](#).

Agenda

- Discuss *unboundedness*; (today)
- Discuss initialization/*infeasibility*; i.e., what if initial dictionary is not feasible. (today)
- Discuss *degeneracy*. (next lecture)

Unboundedness

Consider the following dictionary:

		Current Dictionary								
obj =	0.0	+	2.0	x1	+	-1.0	x2	+	1.0	x3
w1 =	4.0	-	-5.0	x1	-	3.0	x2	-	-1.0	x3
w2 =	10.0	-	-1.0	x1	-	-5.0	x2	-	2.0	x3
w3 =	7.0	-	0.0	x1	-	-4.0	x2	-	3.0	x3
w4 =	6.0	-	-2.0	x1	-	-2.0	x2	-	4.0	x3
w5 =	6.0	-	-3.0	x1	-	0.0	x2	-	-3.0	x3

- Could increase either x_1 or x_3 to increase obj.
- Consider increasing x_1 .
- Which basic variable decreases to zero first?
- Answer: none of them, x_1 can grow without bound, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.

Initialization

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & -3x_1 + 4x_2 \\ \text{subject to} & -4x_1 - 2x_2 \leq -8 \\ & -2x_1 \leq -2 \\ & 3x_1 + 2x_2 \leq 10 \\ & -x_1 + 3x_2 \leq 1 \\ & -3x_2 \leq -2 \\ & x_1, x_2 \geq 0. \end{array}$$

Phase-I Problem

- Modify problem by subtracting a new variable, x_0 , from each constraint and
- replacing objective function with $-x_0$

Phase-I Problem

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & -x_0 - 4x_1 - 2x_2 \leq -8 \\ & -x_0 - 2x_1 \leq -2 \\ & -x_0 + 3x_1 + 2x_2 \leq 10 \\ & -x_0 - x_1 + 3x_2 \leq 1 \\ & -x_0 - 3x_2 \leq -2 \\ & x_0, x_1, x_2 \geq 0. \end{array}$$

- Clearly feasible: pick x_0 large, $x_1 = 0$ and $x_2 = 0$.
- If optimal solution has $\text{obj} = 0$, then original problem is feasible.
- Final phase-I basis can be used as initial *phase-II* basis (ignoring x_0 thereafter).
- If optimal solution has $\text{obj} < 0$, then original problem is infeasible.

Initialization—First Pivot

Applet depiction shows both the Phase-I and the Phase-II objectives:

		Current Dictionary								
obj =	0.0	+	0.0	x0	+	-3.0	x1	+	4.0	x2
	0.0	+	-1.0	x0	+	0.0	x1	+	0.0	x2
w1 =	-8.0	-	-1.0	x0	-	-4.0	x1	-	-2.0	x2
w2 =	-2.0	-	-1.0	x0	-	-2.0	x1	-	0.0	x2
w3 =	10.0	-	-1.0	x0	-	3.0	x1	-	2.0	x2
w4 =	1.0	-	-1.0	x0	-	-1.0	x1	-	3.0	x2
w5 =	-2.0	-	-1.0	x0	-	0.0	x1	-	-3.0	x2

- Dictionary is infeasible even for Phase-I.
- One pivot needed to get feasible.
- Entering variable is x_0 .
- Leaving variable is one whose current value is most negative, i.e. w_1 .
- After first pivot...

Initialization—Second Pivot

Going into second pivot:

		Current Dictionary								
obj =	0.0	+	0.0	w1	+	-3.0	x1	+	4.0	x2
	-8.0	+	-1.0	w1	+	4.0	x1	+	2.0	x2
x0 =	8.0	-	-1.0	w1	-	4.0	x1	-	2.0	x2
w2 =	6.0	-	-1.0	w1	-	2.0	x1	-	2.0	x2
w3 =	18.0	-	-1.0	w1	-	7.0	x1	-	4.0	x2
w4 =	9.0	-	-1.0	w1	-	3.0	x1	-	5.0	x2
w5 =	6.0	-	-1.0	w1	-	4.0	x1	-	-1.0	x2

- Feasible!
- Focus on the yellow highlights.
- Let x_1 enter.
- Then w_5 must leave.
- After second pivot...

Initialization—Third Pivot

Going into third pivot:

Current Dictionary										
obj =	-4.5	+	-0.75	w1	+	0.75	w5	+	3.25	x2
	-2.0	+	0.0	w1	+	-1.0	w5	+	3.0	x2
x0 =	2.0	-	0.0	w1	-	-1.0	w5	-	3.0	x2
w2 =	3.0	-	-0.5	w1	-	-0.5	w5	-	2.5	x2
w3 =	7.5	-	0.75	w1	-	-1.75	w5	-	5.75	x2
w4 =	4.5	-	-0.25	w1	-	-0.75	w5	-	5.75	x2
x1 =	1.5	-	-0.25	w1	-	0.25	w5	-	-0.25	x2

- x_2 must enter.
- x_0 must leave.
- After third pivot...

End of Phase-I

Current dictionary:

Current Dictionary							
obj =	-7/3	+	-3/4	w1 +	11/6	w5 +	0 x0
	0	+	0	w1 +	0	w5 +	0 x0
x2 =	2/3	-	0	w1 -	-1/3	w5 -	0 x0
w2 =	4/3	-	-1/2	w1 -	1/3	w5 -	0 x0
w3 =	11/3	-	3/4	w1 -	1/6	w5 -	0 x0
w4 =	2/3	-	-1/4	w1 -	7/6	w5 -	0 x0
x1 =	5/3	-	-1/4	w1 -	1/6	w5 -	0 x0

- Optimal for Phase-I (no yellow highlights).
- $\text{obj} = 0$, therefore original problem is feasible.

Phase-II

Current dictionary:

Current Dictionary							
obj =	-7/3	+	-3/4	w1 +	11/6	w5 +	0 x0
	0	+	0	w1 +	0	w5 +	0 x0
x2 =	2/3	-	0	w1 -	-1/3	w5 -	0 x0
w2 =	4/3	-	-1/2	w1 -	1/3	w5 -	0 x0
w3 =	11/3	-	3/4	w1 -	1/6	w5 -	0 x0
w4 =	2/3	-	-1/4	w1 -	7/6	w5 -	0 x0
x1 =	5/3	-	-1/4	w1 -	1/6	w5 -	0 x0

For Phase-II:

- Ignore column with x_0 in Phase-II.
- Ignore Phase-I objective row.

w_5 must enter. w_4 must leave...

Optimal Solution

Current Dictionary										
obj =	-9/7	+	-5/14	w1	+	-11/7	w4	+	0	x0
	0	+	0	w1	+	0	w4	+	0	x0
x2 =	6/7	-	-1/14	w1	-	2/7	w4	-	0	x0
w2 =	8/7	-	-3/7	w1	-	-2/7	w4	-	0	x0
w3 =	25/7	-	11/14	w1	-	-1/7	w4	-	0	x0
w5 =	4/7	-	-3/14	w1	-	6/7	w4	-	0	x0
x1 =	11/7	-	-3/14	w1	-	-1/7	w4	-	0	x0

- Optimal!
- Click [here](#) to practice the simplex method on problems that may have infeasible first dictionaries.
- For instructions, click [here](#).