



Linear Programming: Chapter 13

Network Flows: Algorithms

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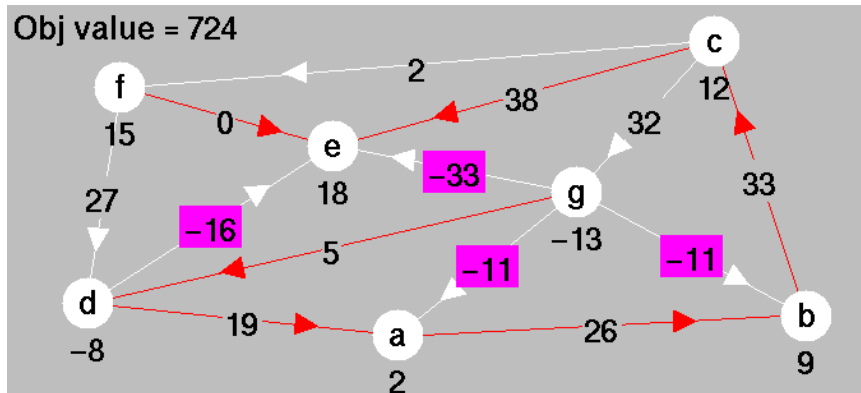
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Agenda

- Primal Network Simplex Method
- Dual Network Simplex Method
- Two-Phase Network Simplex Method
- One-Phase Primal-Dual Network Simplex Method
- Planar Graphs
- Integrality Theorem

Primal Network Simplex Method

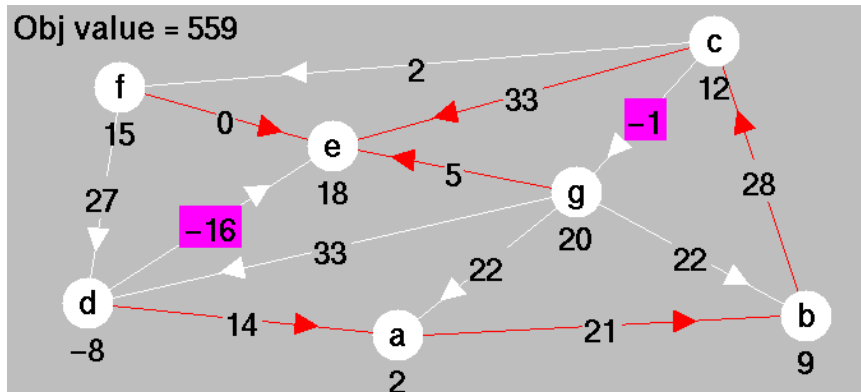
Used when all primal flows are nonnegative (i.e., primal feasible).



Pivot Rules:

Entering arc: Pick a nontree arc having a negative (i.e. infeasible) dual slack.

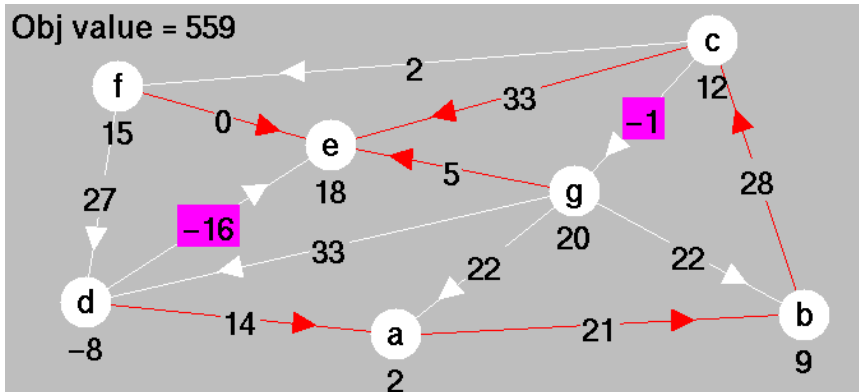
Entering arc: (g,e)
Leaving arc: (g,d)



Leaving arc: Add entering arc to make a *cycle*.

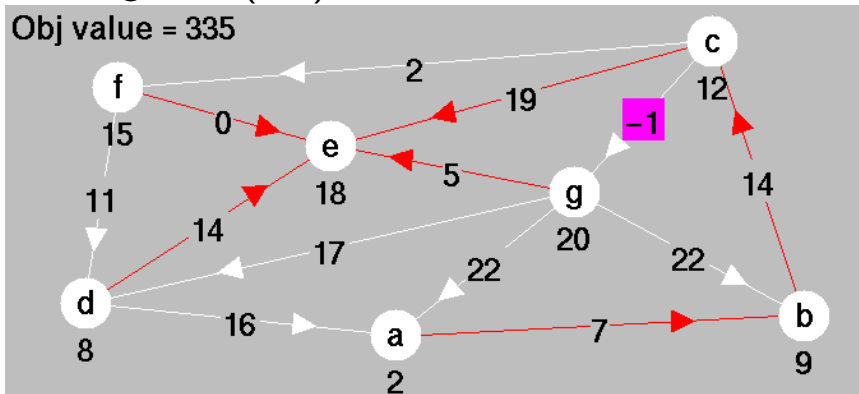
Leaving arc is an arc on the cycle, pointing in the *opposite* direction to the entering arc, and of all such arcs, it is the one with the *smallest* primal flow.

Primal Method—Second Pivot



Entering arc: (d,e)

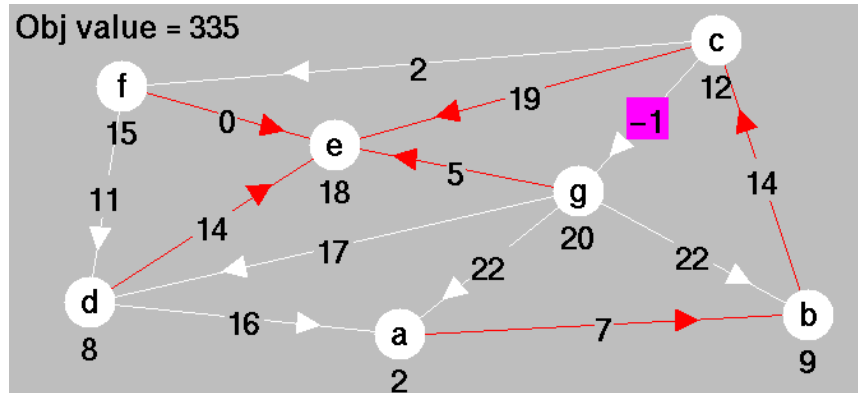
Leaving arc: (d,a)



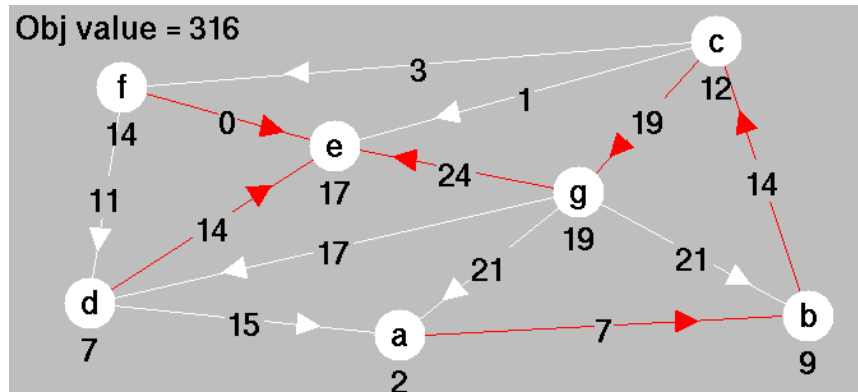
Explanation of leaving arc rule:

- Increase flow on (d,e).
- Each unit increase produces a unit *increase* on arcs pointing in the *same* direction.
- Each unit increase produces a unit *decrease* on arcs pointing in the *opposite* direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.

Primal Method—Third Pivot



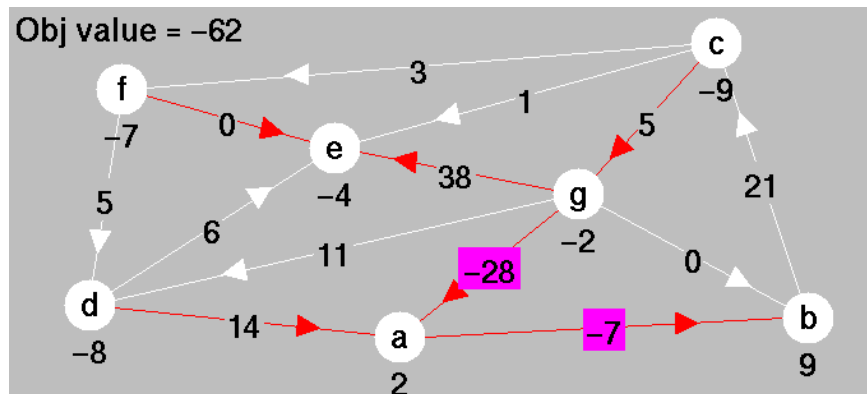
Entering arc: (c,g)
Leaving arc: (c,e)



Optimal!

Dual Network Simplex Method

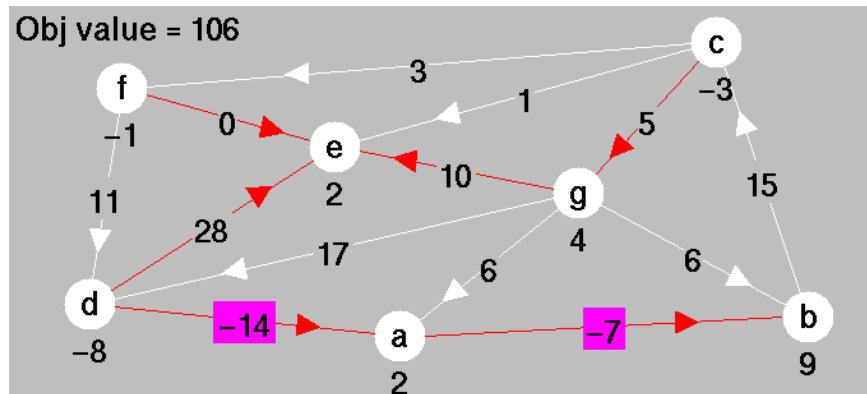
Used when all dual slacks are nonnegative (i.e., dual feasible).



Pivot Rules:

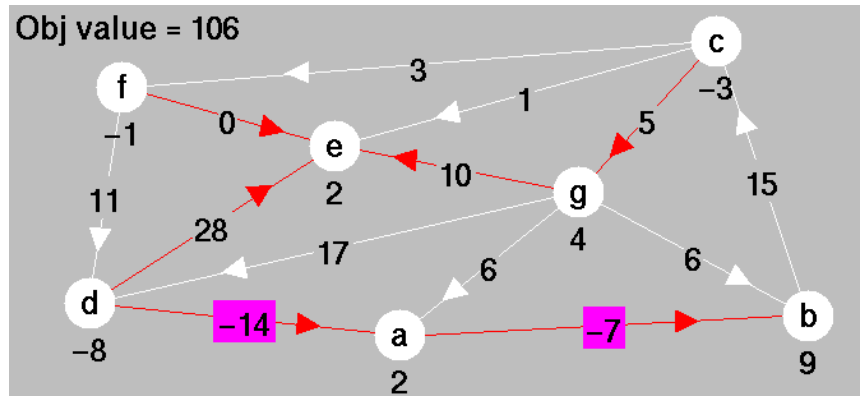
Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.

Leaving arc: (g,a)
 Entering arc: (d,e)

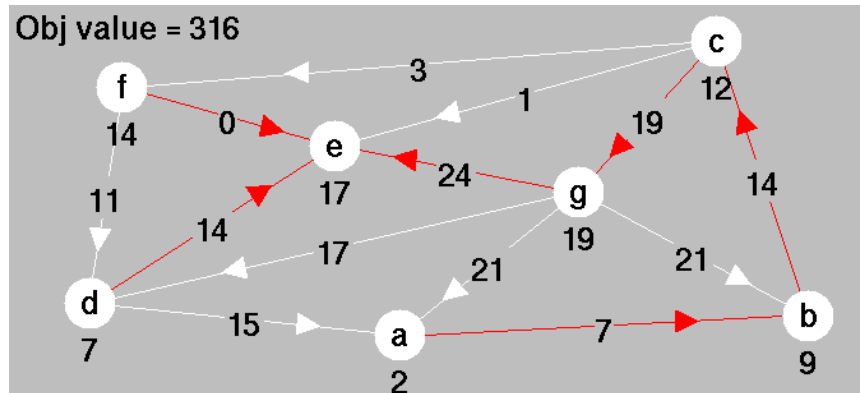


Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the *opposite* direction, and, of all such arcs, is the one with the *smallest* dual slack.

Dual Network Simplex Method—Second Pivot



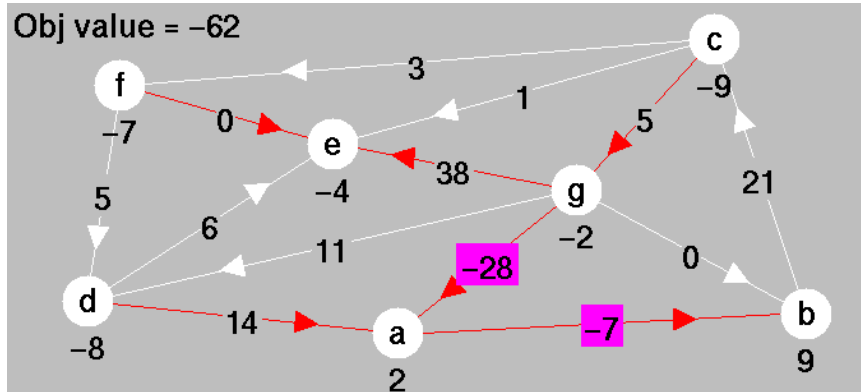
Leaving arc: (d,a)
Entering arc: (b,c)



Optimal!

Explanation of Entering Arc Rule

Recall initial tree solution:

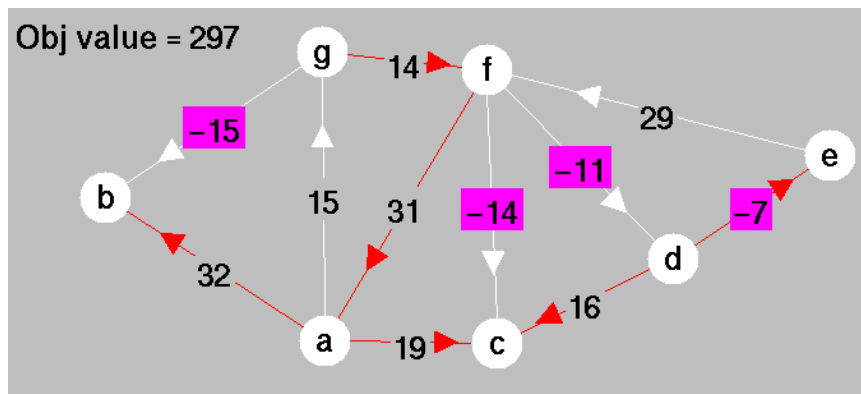


- Remove leaving arc. Need to find a reconnecting arc.
- Consider some reconnecting arc. Add flow to it.
 - If it reconnects in the same direction as leaving arc, such as (f,d), then flow on leaving arc decreases.
 - Therefore, leaving arc's flow can't be *raised* to zero.
 - Therefore, leaving arc can't leave. No good.

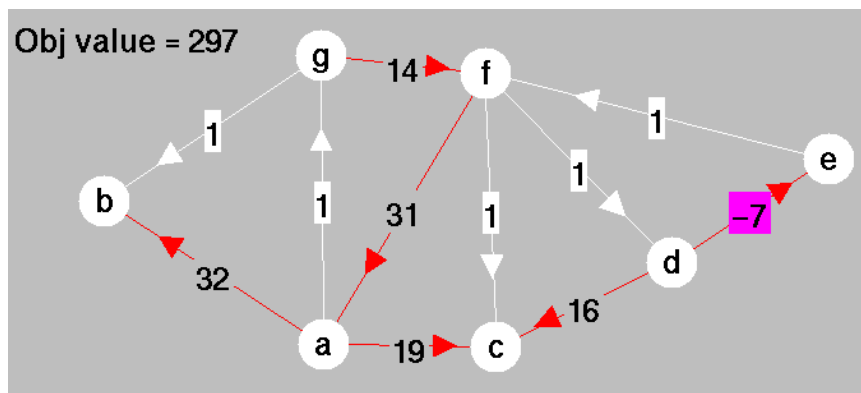
- Consider a potential arc reconnecting in the opposite direction, say (b,c).
 - Its dual slack will drop to zero.
 - All other reconnecting arcs pointing in the same direction will drop by the same amount.
 - To maintain nonnegativity of all the others, must pick the one that drops the least.

Two-Phase Network Simplex Method

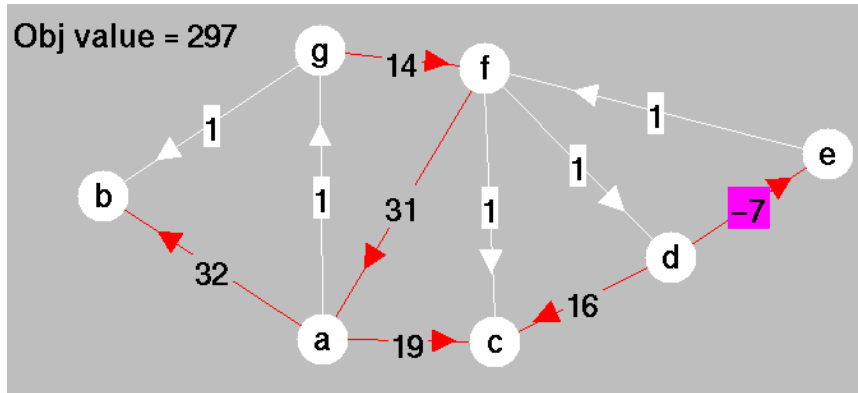
Example.



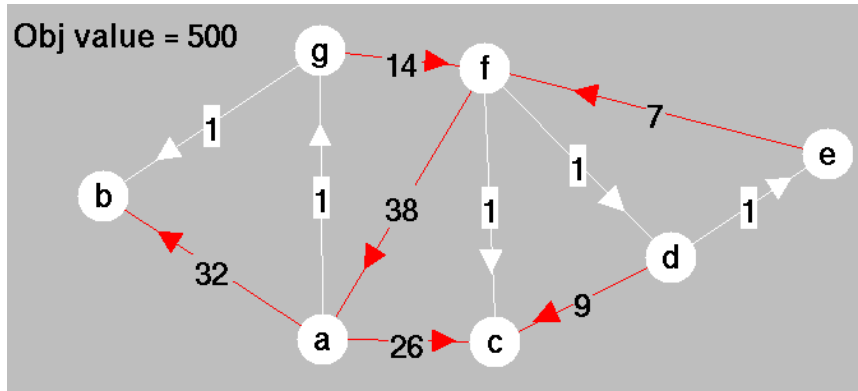
- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.



Two-Phase Method–First Pivot

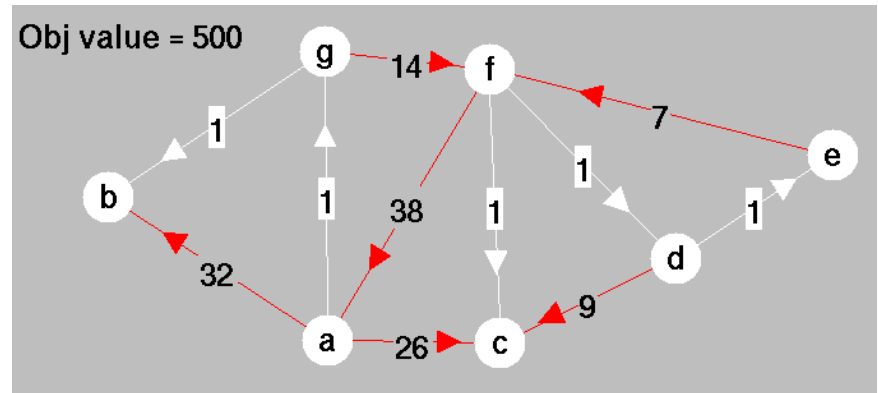


Use dual network simplex method.
Leaving arc: (d,e) Entering arc: (e,f)

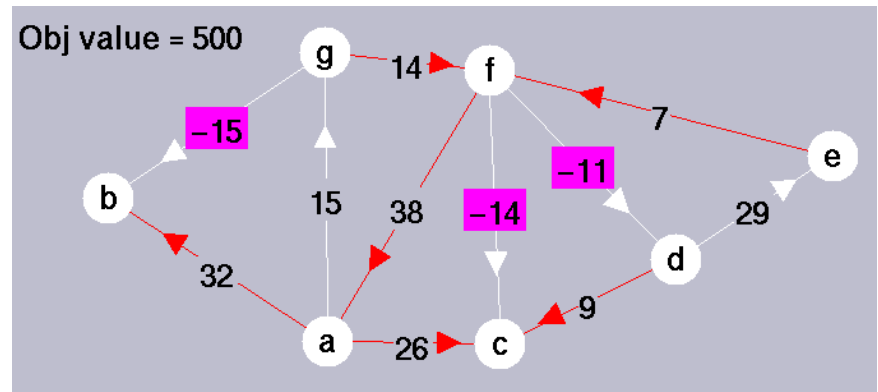


Dual Feasible!

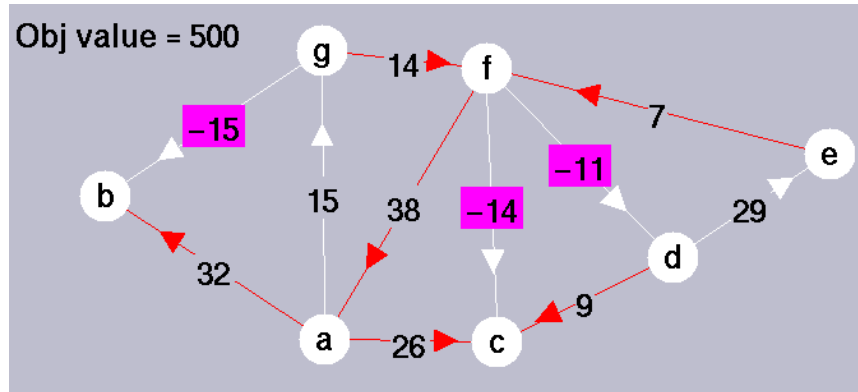
Two-Phase Method–Phase II



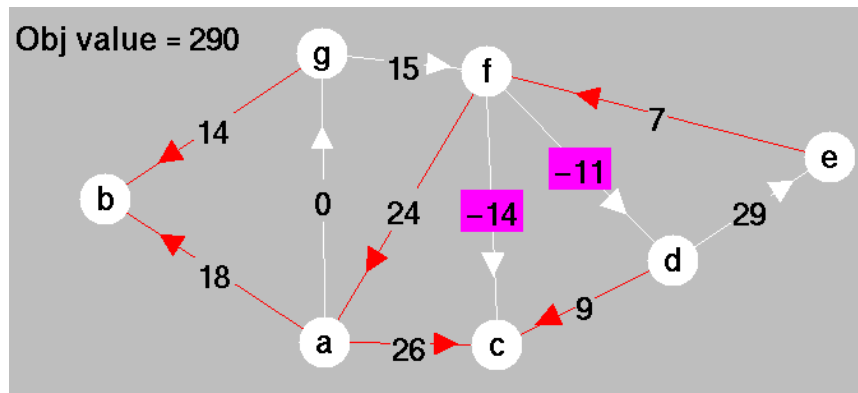
- Turn off display of artificial dual slacks.
- Turn on display of dual slacks.



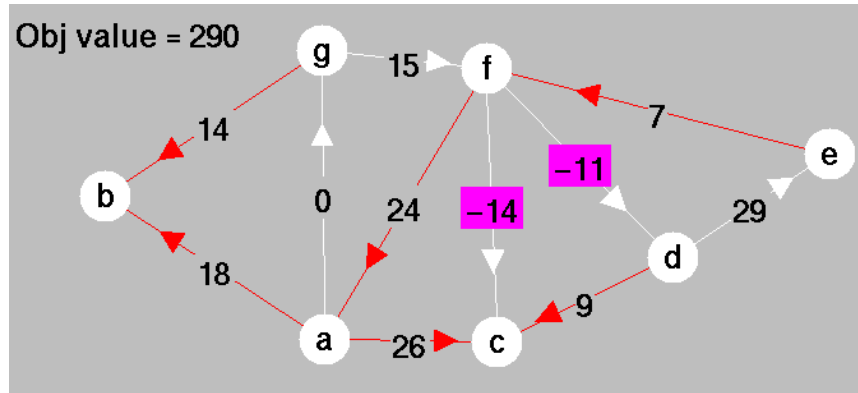
Two-Phase Method–Second Pivot



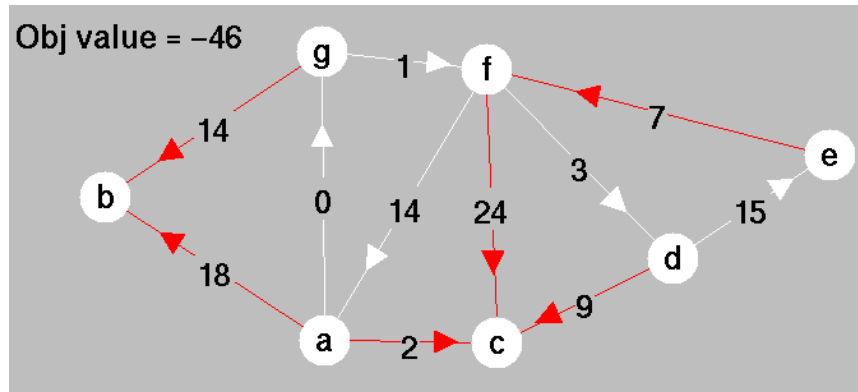
Entering arc: (g,b)
Leaving arc: (g,f)



Two-Phase Method–Third Pivot



Entering arc: (f,c)
Leaving arc: (a,f)



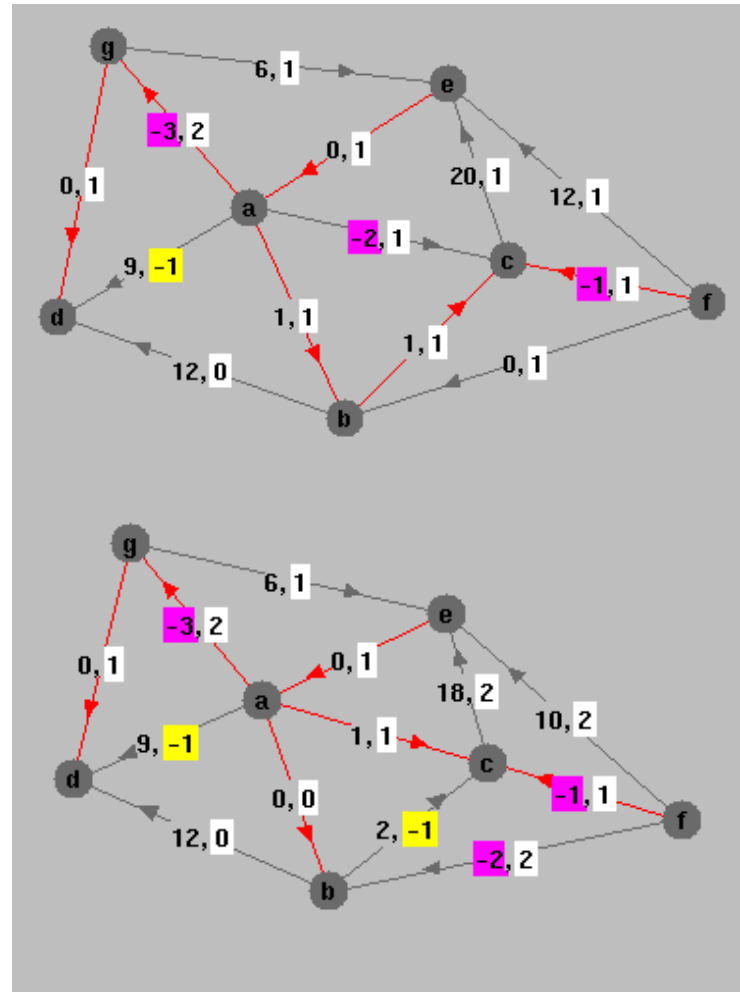
Optimal!

Online Network Simplex Pivot Tool

Click [here](#) (or on any displayed network) to try out the online network simplex pivot tool.

Second Iteration

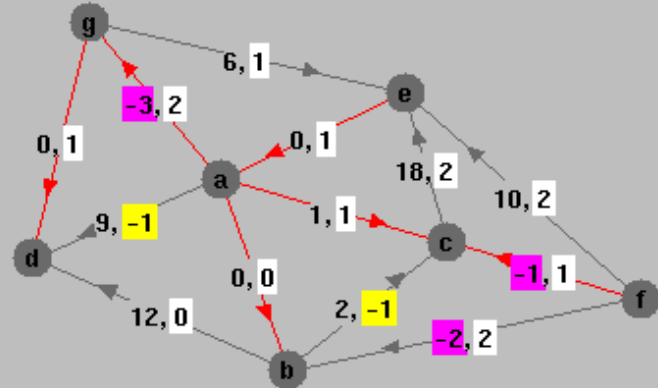
- Range of μ values:
 $2 \leq \mu \leq 9$.
- Entering arc: (a,c)
- Leaving arc: (b,c)



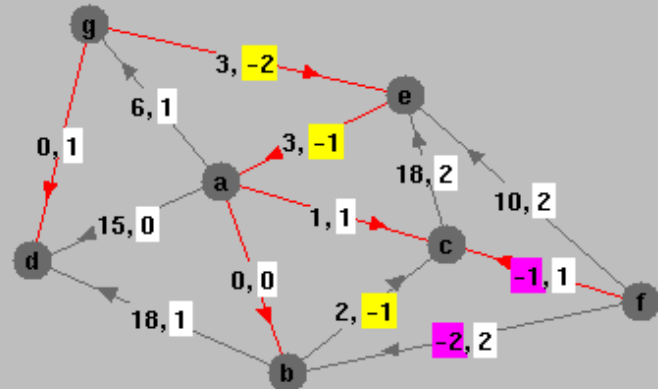
New tree:

Third Iteration

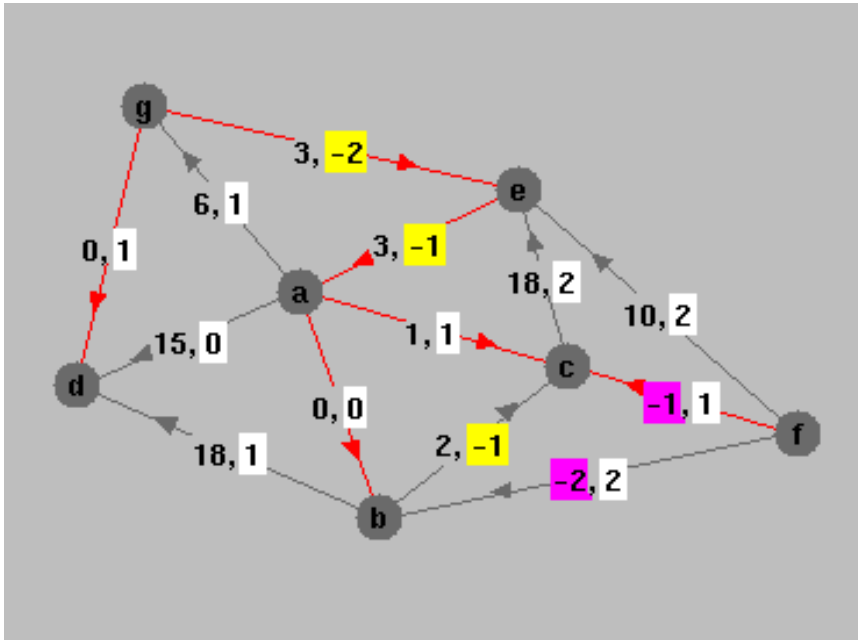
- Range of μ values:
 $1.5 \leq \mu \leq 2$.
- Leaving arc: (a,g)
- Entering arc: (g,e)



New tree:



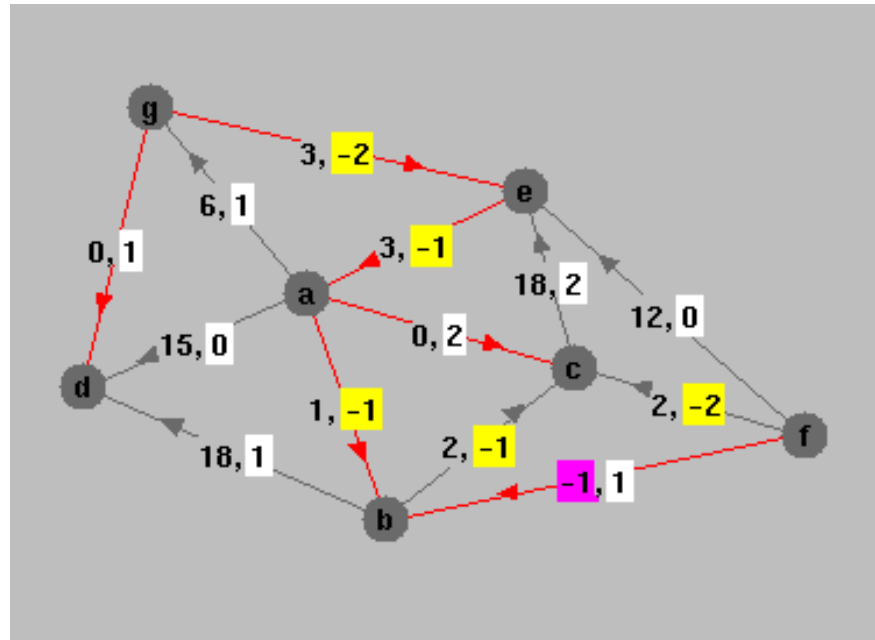
Fourth Iteration



- Range of μ values:
 $1 \leq \mu \leq 1.5$.
- A tie:
 - Arc (f,b) enters, or
 - Arc (f,c) leaves.
- Decide arbitrarily:
 - Leaving arc: (f,c)
 - Entering arc: (f,b)

Fifth Iteration

- Range of μ values: $1 \leq \mu \leq 1$.
- Leaving arc: (f,b)
- Nothing to Enter.



Primal Infeasible!

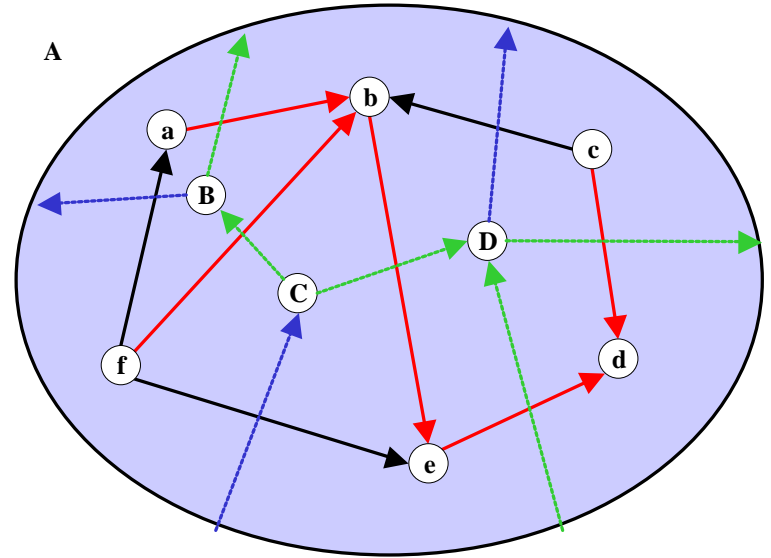
Online Network Simplex Pivot Tool

Click [here](#) (or on any displayed network) to try out the online network simplex pivot tool.

Planar Networks

Definition. Network is called **planar** if can be drawn on a plane without intersecting arcs.

Theorem. Every planar network has a dual—dual nodes are **faces** of primal network.



Notes:

- Dual node A is “node at infinity”.
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. A dual pivot on the primal network is exactly a primal pivot on the dual network.

Integrality Theorem

Theorem. *Assuming integer data, every basic feasible solution assigns integer flow to every arc.*

Corollary. *Assuming integer data, every basic optimal solution assigns integer flow to every arc.*