



Linear Programming: Chapter 13

Network Flows: Theory

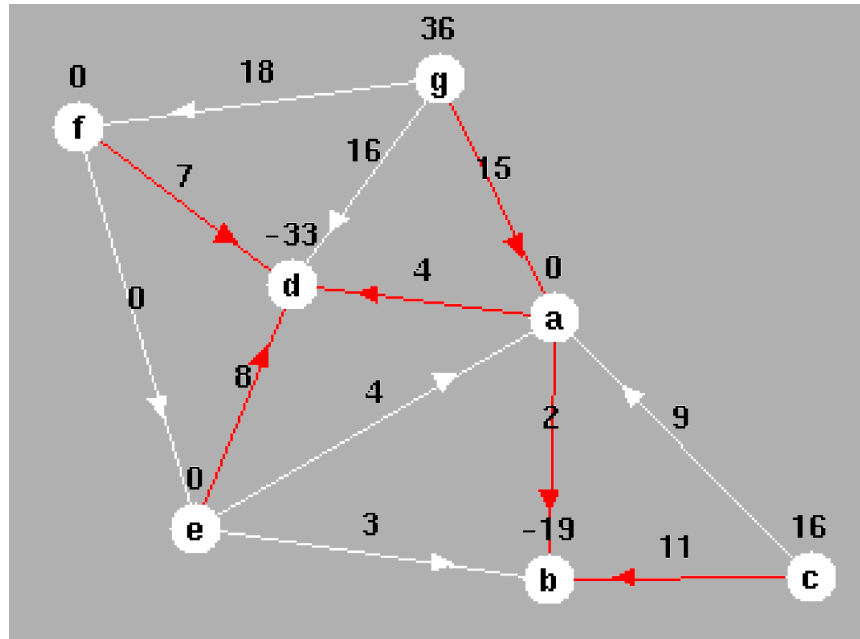
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October 19, 2010

Slides last edited on October 19, 2010

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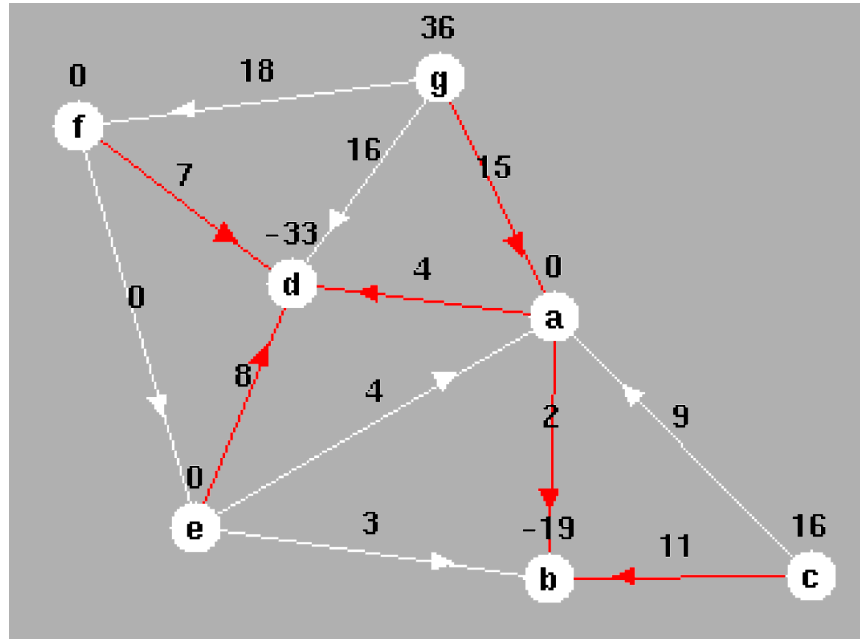
Networks



Basic elements:

- \mathcal{N} *Nodes* (let m denote number of them).
- \mathcal{A} Directed *Arcs*
 - subset of all possible arcs: $\{(i, j) : i, j \in \mathcal{N}, i \neq j\}$.
 - arcs are *directed*: $(i, j) \neq (j, i)$.

Network Flow Data



- b_i , $i \in \mathcal{N}$, *supply* at node i
- c_{ij} , $(i, j) \in \mathcal{A}$, *cost* of shipping 1 unit along arc (i, j) .

Note: *demands* are recorded as *negative supplies*.

Network Flow Problem

Decision Variables:

x_{ij} , $(i, j) \in \mathcal{A}$, *quantity* to ship along arc (i, j) .

Objective:

$$\text{minimize } \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

Network Flow Problem–Cont.

Constraints:

- Mass conservation (aka flow balance):

$$\begin{aligned} \text{inflow}(k) - \text{outflow}(k) &= \text{demand}(k) = -\text{supply}(k), & k \in \mathcal{N} \\ &\Downarrow \\ \sum_{\substack{i: \\ (i,k) \in \mathcal{A}}} x_{ik} - \sum_{\substack{j: \\ (k,j) \in \mathcal{A}}} x_{kj} &= -b_k, & k \in \mathcal{N} \end{aligned}$$

- Nonnegativity:

$$x_{ij} \geq 0, \quad (i,j) \in \mathcal{A}$$

Dual Problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + z = c \\ & z \geq 0 \end{array}$$

In network notation:

$$\begin{array}{ll} \text{maximize} & -\sum_{i \in \mathcal{N}} b_i y_i \\ \text{subject to} & y_j - y_i + z_{ij} = c_{ij} \quad (i, j) \in \mathcal{A} \\ & z_{ij} \geq 0 \quad (i, j) \in \mathcal{A} \end{array}$$

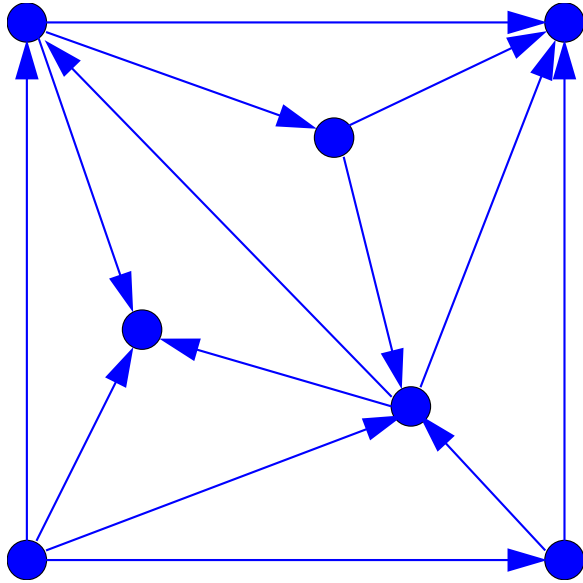
Complementarity Relations

- The primal variables must be nonnegative.
- Therefore the associated dual constraints are inequalities.
- The dual slack variables are complementary to the primal variables:

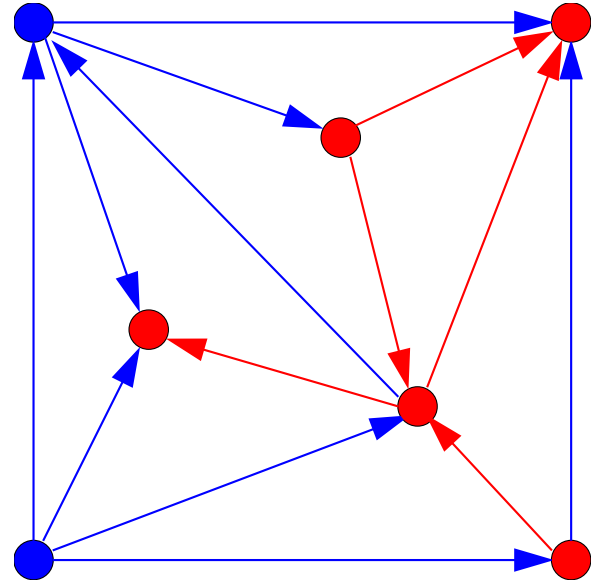
$$x_{ij}z_{ij} = 0, \quad (i, j) \in \mathcal{A}$$

- The primal constraints are equalities.
- Therefore they have no slack variables.
- The corresponding dual variables, the y_i 's, are free variables.
- No complementarity conditions apply to them.

Definition: Subnetwork

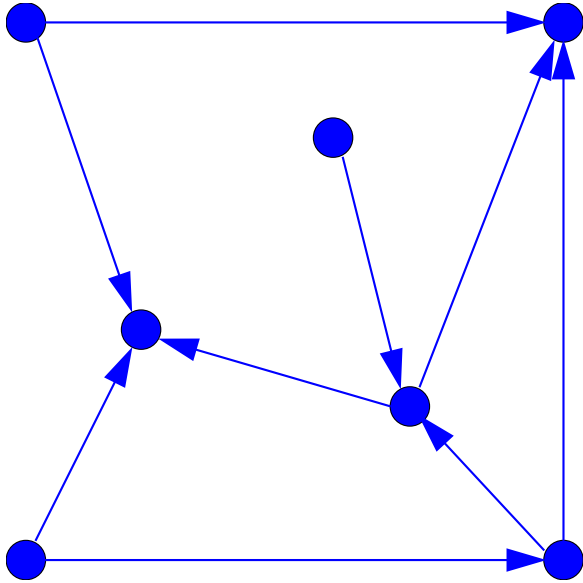


Network

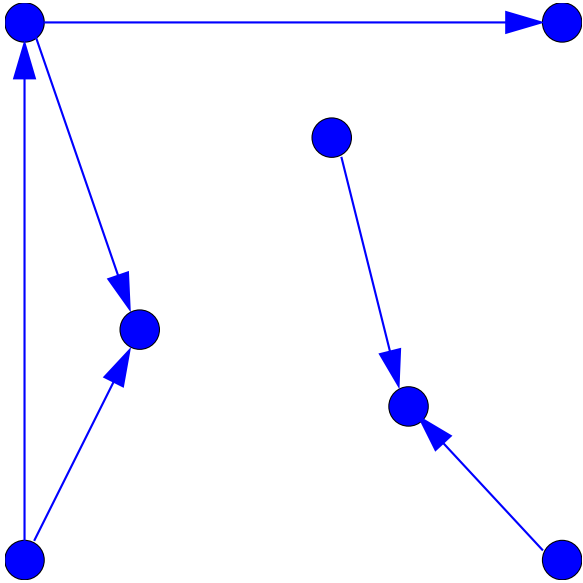


Subnetwork

Connected vs. Disconnected

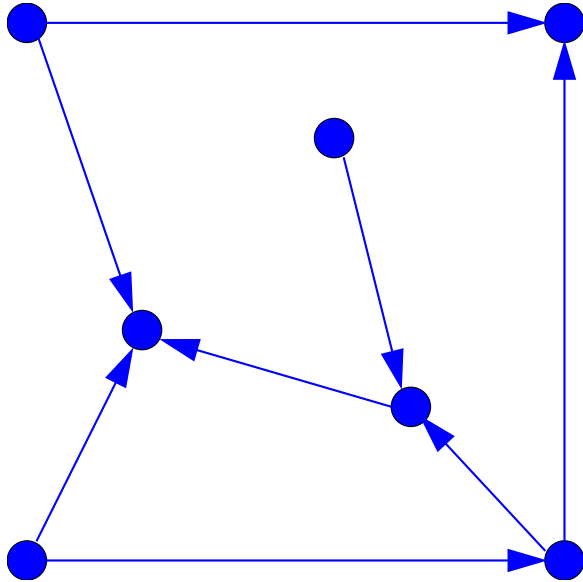


Connected

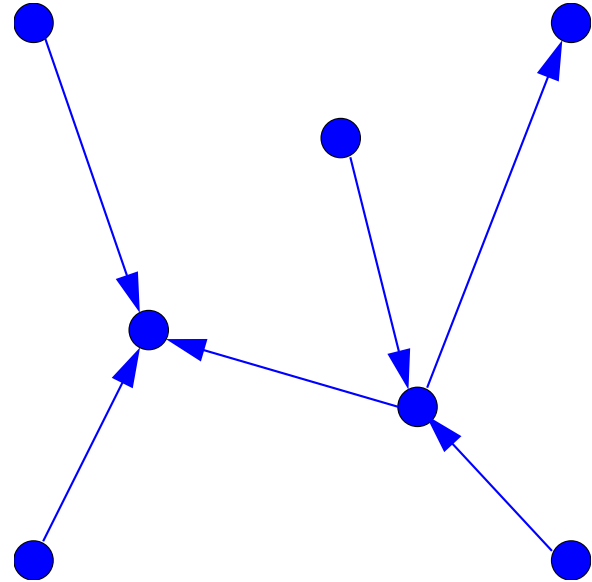


Disconnected

Cyclic vs. Acyclic

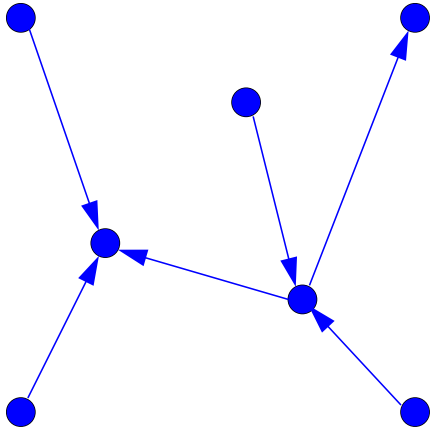


Cyclic

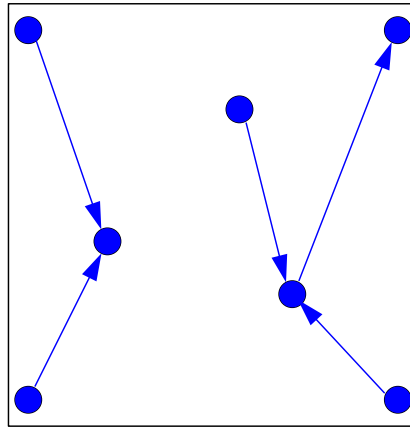


Acyclic

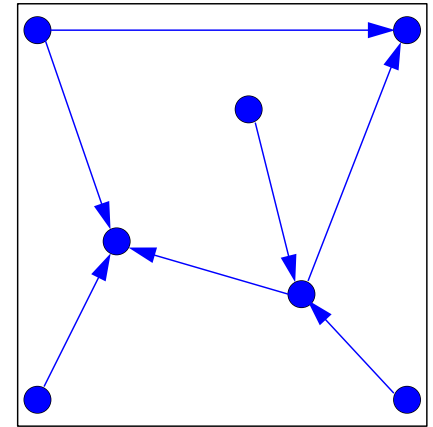
Trees



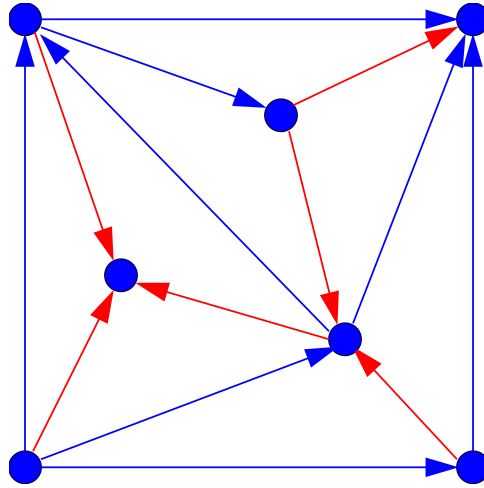
Tree = Connected + Acyclic



Not Trees



Spanning Trees



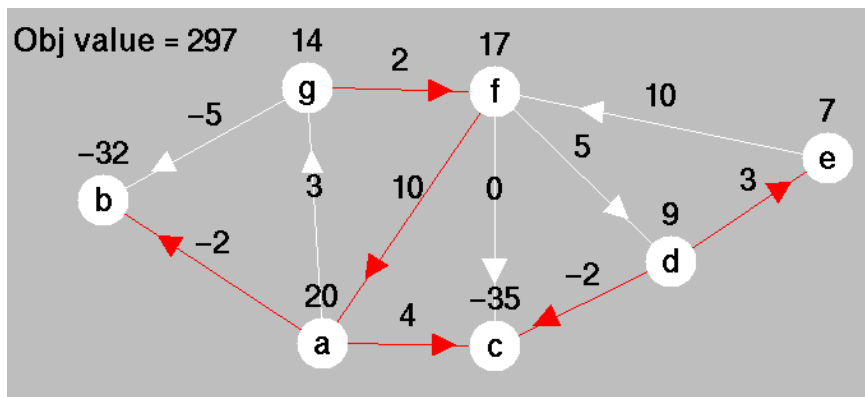
Spanning Tree—A tree touching every node

Tree Solution

$$x_{ij} = 0 \quad \text{for } (i, j) \notin \text{Tree Arcs}$$

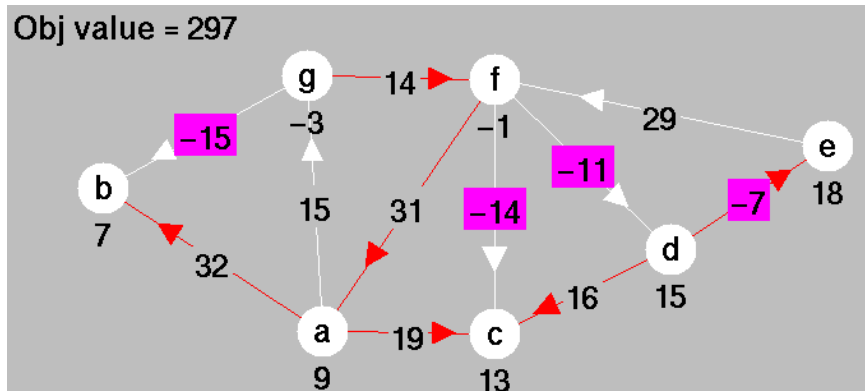
Note: Tree solutions are easy to compute—start at the leaves and work inward...

Online Pivot Tool–Notations



Data:

- *Costs* on arcs shown above arcs.
- *Supplies* at nodes shown above nodes.

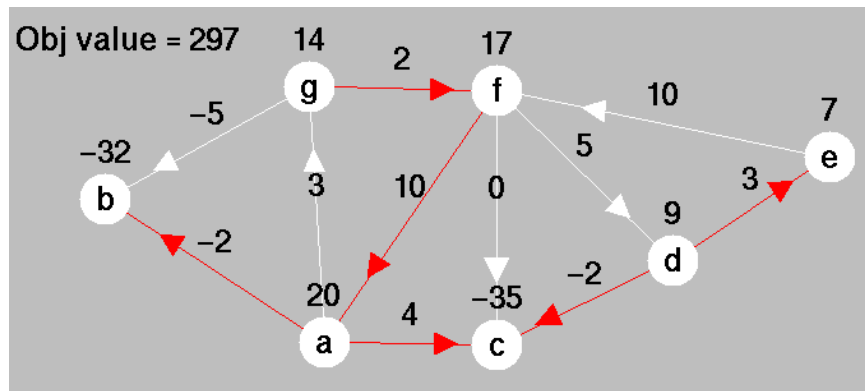


Variables:

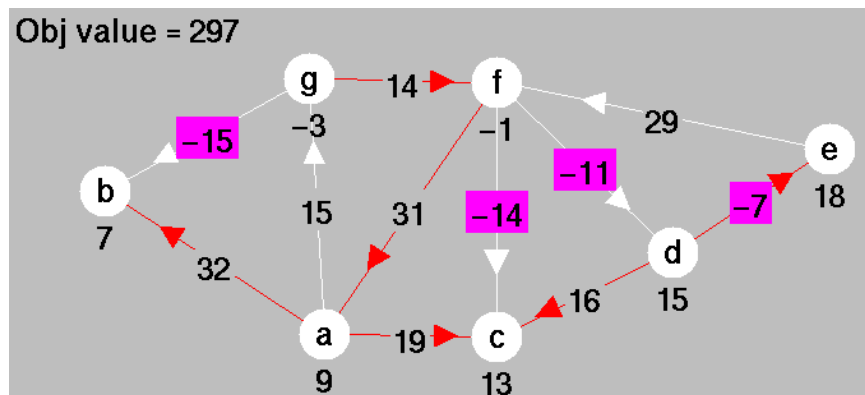
- *Primal flows* shown on tree arcs.
- *Dual slacks* shown on nontree arcs.
- *Dual variables* shown below nodes.

Tree Solutions—An Example

Data:



Variables:



- Fix a root node, say a.
- *Primal flows* on tree arcs calculated recursively from leaves inward.
- *Dual variables* at nodes calculated recursively from root node outward along tree arcs using:

$$y_j - y_i = c_{ij}$$
- *Dual slacks* on nontree arcs calculated using:

$$z_{ij} = y_i - y_j + c_{ij}.$$