Linear Programming: Chapter 1
Introduction

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Resource Allocation

$$
\begin{array}{lrcrcrcl}
\text{maximize} & c_1 x_1 & + & c_2 x_2 & + \cdots + & c_n x_n & & \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + \cdots + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + \cdots + & a_{2n} x_n & \leq & b_2 \\
 & & & & \vdots & & & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + \cdots + & a_{mn} x_n & \leq & b_m \\
 & & & & x_1, \; x_2, \ldots, & x_n & \geq & 0,
\end{array}
$$

where

$$
\begin{aligned}
c_j &= \text{profit per unit of product } j \text{ produced} \\
b_i &= \text{units of raw material } i \text{ on hand} \\
a_{ij} &= \text{units of raw material } i \text{ required to produce one unit of product } j.
\end{aligned}
$$

Blending Problems (Diet Problem)

$$
\begin{array}{lccccccccc}
\text{minimize} & & & c_1x_1 & + & c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{subject to} & l_1 & \leq & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & u_1 \\
 & l_2 & \leq & a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & \leq & u_2 \\
 & & & & & & \vdots & & & \\
 & l_m & \leq & a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & u_m \\
 & & & x_1, & x_2, \ldots, & x_n & \geq & 0 & , &
\end{array}
$$

where

$$
\begin{aligned}
c_j &= \text{cost per unit of food } j \\
l_i &= \text{minimum daily allowance of nutrient } i \\
u_i &= \text{maximum daily allowance of nutrient } i \\
a_{ij} &= \text{units of nutrient } i \text{ contained in one unit of food } j.
\end{aligned}
$$

Fairness in Grading[a]

	MAT	CHE	ANT	REL	POL	ECO	GPA
John			C+	B−	B	B+	2.83
Paul		C+	B−		B+	A−	3.00
George	C+	B−		B+	A−		3.00
Ringo	B−	B	B+	A−			3.18
Avg.	2.5	2.7	2.77	3.23	3.33	3.5	

The Model

Paul got a B+ (3.3) in Politics.

We wish to assert that Paul's actual grade plus a measure of the level of difficulty in Politics courses equals Paul's aptitude plus some small error:

$$\text{Paul's grade in Politics} + \text{Difficulty of Politics} = \text{Paul's Aptitude} + \text{error term}$$

Try to find numerical values for *Aptitudes* and *Difficulties* by minimizing the sum of the error terms over all student/course grades.

[a] All characters appearing herein are fictitious. Any resemblance to real persons, living or dead, is purely coincidental.

Minimizing The Sum Of The Errors

We don't want negative errors to cancel with positive errors.

We could minimize the sum of the squares of the errors (*least squares*).

Or, we could minimize the sum of the absolute values of the errors (*least absolute deviations*).

I used the latter—it provides answers that are analogous to *medians* rather than simple averages (means).

Fixing a Point of Reference

The (course-enrollment weighted) sum of difficulties is constrained to be zero.

The Model

We assume that every grade, $g_{i,j}$ for student i in course j, can be decomposed as a difference between

1. *aptitude*, a_i, of student i, and
2. *difficulty*, d_j, of course j,
3. plus some small correction $\varepsilon_{i,j}$.

That is,

$$g_{i,j} = a_i - d_j + \varepsilon_{i,j}.$$

The $g_{i,j}$'s are data. We wish to find the a_i's and the d_j's that minimizes the sum of the absolute values of the $\varepsilon_{i,j}$'s:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i,j} |\varepsilon_{i,j}| & \\ \text{subject to} & g_{i,j} = a_i - d_j + \varepsilon_{i,j} & \text{for student-course pairs } (i,j) \\ & \displaystyle\sum_j d_j = 0. & \end{array}$$

Absolute Value Trick

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} |\varepsilon_{i,j}| \\ \text{subject to} \quad & g_{i,j} = a_i - d_j + \varepsilon_{i,j} \qquad \text{for all students } i \text{ and all courses } j \\ & \sum_j d_j = 0. \end{aligned}$$

is equivalent to

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} t_{i,j} \\ \text{subject to} \quad & g_{i,j} - a_i + d_j \leq t_{i,j} \qquad \text{for all students } i \text{ and all courses } j \\ & -t_{i,j} \leq g_{i,j} - a_i + d_j \qquad \text{for all students } i \text{ and all courses } j \\ & \sum_j d_j = 0. \end{aligned}$$

The AMPL Model

```
set STUDS;
set COURSES;
set GRADES within {STUDS, COURSES};

param grade {GRADES};

var aptitude {STUDS};
var difficulty {COURSES};
var dev {GRADES} >= 0;

minimize sum_dev: sum {(s,c) in GRADES} dev[s,c];

subject to def_pos_dev {(s,c) in GRADES}:   aptitude[s] - difficulty[c] - grade[s,c] <= dev[s,c];

subject to def_neg_dev {(s,c) in GRADES}:   -dev[s,c] <= aptitude[s] - difficulty[c] - grade[s,c];

subject to normalized_difficulty:   sum {c in COURSES} difficulty[c] = 0;

data;
set STUDS := include "names" ;
set COURSES := include "courses" ;
param: GRADES: grade := include "namecoursegrade" ;

solve;
```

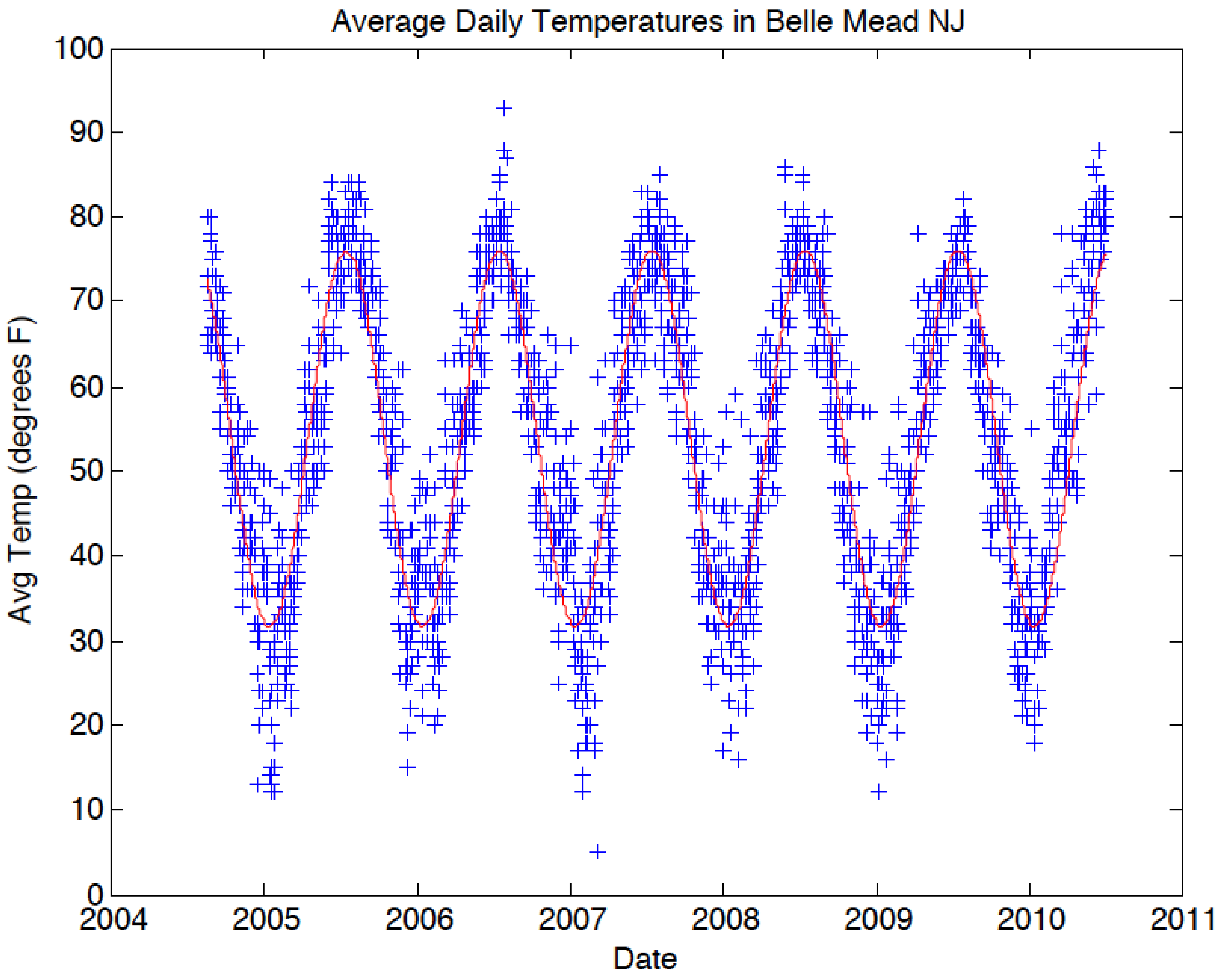
Ringo Is Smarter Than You Thought

	MAT	CHE	ANT	REL	POL	ECO	GPA	Aptitude
John			C+	B−	B	B+	2.83	2.49
Paul		C+	B−		B+	A−	3.00	2.84
George	C+	B−		B+	A−		3.00	3.16
Ringo	B−	B	B+	A−			3.18	3.51
Avg.	2.5	2.7	2.77	3.23	3.33	3.5		
Difficulty	0.84	0.51	0.19	−0.19	−0.51	−0.84		

$$\text{Paul's grade in Politics} + \text{Difficulty of Politics} = \text{Paul's Aptitude} + \text{error term}$$

$$3.3 + (-0.51) = 2.84 + (-0.05)$$

Local Warming



The Temperature Model

Assume daily average temperature has a sinusoidal annual variation superimposed on a linear trend:

$$\text{minimize}_{a_0,\ldots,a_3} \; \sum_{d\in D} \left| a_0 + a_1 d + a_2 \cos(2\pi d/365.25) + a_3 \sin(2\pi d/365.25) - \text{avg}_d \right|$$

Reformulate as a Linear Programming Model

Use the same absolute-value trick again:

$$\begin{aligned}
\text{minimize} \quad & \sum_{d\in D} t_d \\
\text{subject to} \quad & -t_d \le a_0 + a_1 d + a_2 \cos(2\pi d/365.25) + a_3 \sin(2\pi d/365.25) - \text{avg}_d \\
& a_0 + a_1 d + a_2 \cos(2\pi d/365.25) + a_3 \sin(2\pi d/365.25) - \text{avg}_d \le t_d
\end{aligned}$$

AMPL Model

```
set DATES ordered;

param hi {DATES};
param avg {DATES};
param lo {DATES};
param pi := 4*atan(1);

var a {j in 0..3};
var dev {DATES} >= 0, := 1;

minimize sumdev: sum {d in DATES} dev[d];

subject to def_pos_dev {d in DATES}:
    a[0] + a[1]*ord(d,DATES) + a[2]*cos( 2*pi*ord(d,DATES)/365.25)
                             + a[3]*sin( 2*pi*ord(d,DATES)/365.25) - avg[d]
    <= dev[d];
subject to def_neg_dev {d in DATES}:
    -dev[d] <=
    a[0] + a[1]*ord(d,DATES) + a[2]*cos( 2*pi*ord(d,DATES)/365.25)
                             + a[3]*sin( 2*pi*ord(d,DATES)/365.25) - avg[d];

data;
set DATES := include "dates.txt";
param: hi avg lo := include "WXDailyHistory.txt";

solve;

display a;
display a[1]*365.25;
```

It's Getting Warmer in NJ

a[1]*365.25 = 0.0200462

A better model using 55 years of data from McGuire AFB here in NJ is described at

http://www.princeton.edu/~rvdb/ampl/nlmodels/LocalWarming/McGuireAFB/McGuire.html

Portfolio Optimization

Markowitz Shares the 1990 Nobel Prize

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households' and firms' allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

Historical Data

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Notation: $R_j(t)$ = return on investment j in time period t.

Risk vs. Reward

Reward—estimated using historical means:

$$\text{reward}_j = \frac{1}{T}\sum_{t=1}^{T} R_j(t).$$

Risk—Markowitz defined risk as the variability of the returns as measured by the historical variances:

$$\text{risk}_j = \frac{1}{T}\sum_{t=1}^{T} \left(R_j(t) - \text{reward}_j\right)^2.$$

However, to get a linear programming problem (and for other reasons[a]) we use the sum of the absolute values instead of the sum of the squares:

$$\text{risk}_j = \frac{1}{T}\sum_{t=1}^{T} \left|R_j(t) - \text{reward}_j\right|.$$

[a]See http://www.princeton.edu/~rvdb/tex/lpport/lpport8.pdf

Hedging

Investment A: up 20%, down 10%, equally likely—a risky asset.

Investment B: up 20%, down 10%, equally likely—another risky asset.

Correlation: up years for A are down years for B and vice versa.

Portfolio—half in A, half in B: up 5% every year! No risk!

Portfolios

Fractions: x_j = fraction of portfolio to invest in j.

Portfolio's Historical Returns:

$$R(t) = \sum_j x_j R_j(t)$$

Portfolio's Reward:

$$\text{reward}(x) = \frac{1}{T}\sum_{t=1}^{T} R(t) = \frac{1}{T}\sum_{t=1}^{T}\sum_j x_j R_j(t)$$

Portfolio's Risk:

$$
\begin{aligned}
\mathrm{risk}(x) &= \frac{1}{T}\sum_{t=1}^{T}\left|R(t) - \mathrm{reward}(x)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j R_j(t) - \frac{1}{T}\sum_{s=1}^{T}\sum_{j} x_j R_j(s)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j \left(R_j(t) - \frac{1}{T}\sum_{s=1}^{T} R_j(s)\right)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j (R_j(t) - \mathrm{reward}_j)\right|
\end{aligned}
$$

A Markowitz-Type Model

Decision Variables: the fractions x_j.

Objective: maximize return, minimize risk.

Fundamental Lesson: can't simultaneously optimize two objectives.

Compromise: set an upper bound μ for risk and maximize reward subject to this bound constraint:

- Parameter μ is called the risk aversion parameter.
- Large value for μ puts emphasis on reward maximization.
- Small value for μ puts emphasis on risk minimization.

Constraints:

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j x_j(R_j(t) - \text{reward}_j)\right| &\leq \mu \\ \sum_j x_j &= 1 \\ x_j &\geq 0 \qquad \text{for all } j \end{aligned}$$

Optimization Problem

$$\begin{aligned} \text{maximize} \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) \\ \text{subject to} \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j (R_j(t) - \text{reward}_j)\right| \le \mu \\ & \sum_{j} x_j = 1 \\ & x_j \ge 0 \qquad \text{for all } j \end{aligned}$$

Because of absolute values not a linear programming problem.

Easy to convert (as we've already seen)...

A Linear Programming Formulation

$$
\begin{aligned}
\text{maximize} \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) \\
\text{subject to} \quad & -y_t \le \sum_{j} x_j (R_j(t) - \text{reward}_j) \le y_t \qquad \text{for all } t \\
& \frac{1}{T}\sum_{t=1}^{T} y_t \le \mu \\
& \sum_{j} x_j = 1 \\
& x_j \ge 0 \qquad \text{for all } j
\end{aligned}
$$

Efficient Frontier

Varying risk bound μ produces the so-called *efficient frontier*.
Portfolios on the efficient frontier are reasonable.
Portfolios not on the efficient frontier can be strictly improved.

μ	US 3-Month T-Bills	Lehman Bros. Corp. Bonds	NASDAQ Comp.	Wilshire 5000	Gold	EAFE	Reward	Risk
0.1800					0.017	0.983	1.141	0.180
0.1538					0.191	0.809	1.139	0.154
0.1275				0.119	0.321	0.560	1.135	0.128
0.1013				0.407	0.355	0.238	1.130	0.101
0.0751			0.340	0.180	0.260	0.220	1.118	0.075
0.0488	0.172	0.492			0.144	0.008	1.104	0.049
0.0226	0.815	0.100	0.037		0.041	0.008	1.084	0.022

Efficient Frontier