Princeton University
Department of Operations Research
and Financial Engineering

ORF 307
Optimization

Practice Midterm 1

Closed book. No computers. Calculators allowed (but not needed).

You are permitted to use a one-page two-sided cheat sheet.

Please return the exam questions and your cheat sheet with your exam booklet.

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(1) *(30 pts.)* Consider the following problem:

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & 2x_1 & - & x_2 & - & 6x_3 & + & x_4 & & \\
\text{subject to} & x_1 & & & & & & & \le & 4 \\
 & & & -x_2 & & & + & x_4 & \le & 5 \\
 & -x_1 & & & - & x_3 & + & x_4 & \le & -1 \\
 & & & & & x_1,\ x_2, & x_3, & x_4 & \ge & 0.
\end{array}
$$

(a) Write the problem in dictionary form.
(b) Write down the values of all variables associated with this dictionary.
(c) Write down the dual problem as a minimization problem.
(d) Write the dual problem in dictionary form.
(e) Write down the values of all variables associated with this dual dictionary.
(f) Check whether the primal values from (b) are complementary to the dual values from (e).

See next page

1. (a)

$$
\begin{array}{rcl}
\zeta & = & 2x_1 - x_2 - 6x_3 + x_4 \\
\hline
w_1 & = & 4 - x_1 \\
w_2 & = & 5 + x_2 - x_4 \\
w_3 & = & -1 + x_1 + x_3 - x_4
\end{array}
$$

(b) $x_1 = x_2 = x_3 = x_4 = 0$

$w_1 = 4$
$w_2 = 5$
$w_3 = -1$

(c) minimize $4y_1 + 5y_2 - y_3$

$$
\begin{array}{rcl}
y_1 - y_3 & \geq & 2 \\
-y_2 & \geq & -1 \\
-y_3 & \geq & -6 \\
y_2 + y_3 & \geq & 1 \\
y_1, y_2, y_3 & \geq & 0
\end{array}
$$

(d)

$$
\begin{array}{rcl}
-\xi & = & -4y_1 - 5y_2 + y_3 \\
\hline
z_1 & = & -2 + y_1 - y_3 \\
z_2 & = & 1 - y_2 \\
z_3 & = & 6 - y_3 \\
z_4 & = & -1 + y_2 + y_3
\end{array}
$$

(e) $y_1 = y_2 = y_3 = 0$, $z_1 = -2$, $z_2 = 1$, $z_3 = 6$, $z_4 = -1$

(f) $x_1 z_1 = 0 \cdot (-2) = 0$, $x_2 z_2 = 0(1) = 0$, $x_3 z_3 = 0 \cdot 6 = 0$, $x_4 z_4 = 0(-1) = 0$
$w_1 y_1 = 4 \cdot 0 = 0$, $w_2 y_2 = 5 \cdot 0 = 0$, $w_3 y_3 = -1 \cdot 0 = 0$ YES!

(2) *(14 pts.)* Consider the following four dictionaries:

$$
\text{(a)}\quad
\begin{array}{rcl}
\zeta & = & -2 - 2x_1 - 3w_1 - w_2 \\ \hline
x_2 & = & 2 \phantom{{}+2x_1} + 3w_1 - w_2 \\
x_3 & = & 5 + 2x_1 \phantom{{}+3w_1} - w_2
\end{array}
\qquad
\text{(b)}\quad
\begin{array}{rcl}
\zeta & = & 3 + w_1 + 2x_2 - 2w_2 - x_3 \\ \hline
x_1 & = & 4 + 6x_2 + w_2 - 3x_3 \\
x_4 & = & 3 - 4w_1 - x_3
\end{array}
$$

$$
\text{(c)}\quad
\begin{array}{rcl}
\zeta & = & -9 - 3x_2 \\ \hline
x_3 & = & -2 - 2w_2 - w_3 - x_2 \\
w_1 & = & -1 + w_2 + 5w_3 + x_2 \\
x_1 & = & 3 - w_2 - 3w_3 - x_2
\end{array}
\qquad
\text{(d)}\quad
\begin{array}{rcl}
\zeta & = & 6 + 2x_4 + x_1 + 2w_3 \\ \hline
x_3 & = & -1 + 2x_4 + w_1 + 3x_1 - 7w_3 \\
x_2 & = & -2 - x_4 + 5w_1 + x_1 + w_3 \\
w_2 & = & - x_4 - 3w_1 - 2x_1 + w_3
\end{array}
$$

For each of the above dictionaries, identify which of the following statements are true by placing check marks in the following table:

	(a)	(b)	(c)	(d)
The associated primal solution is feasible	X	X		
The associated primal solution is optimal	X			
The dictionary is primal degenerate				X
The dictionary shows the primal problem is unbounded		X		
Variable x_3 is basic	X		X	X
The associated dual solution is dual feasible	X		X	
The dictionary shows the primal problem is infeasible			X	

Be sure to return this page with your exam booklet.

(3) *(18 pts.)* Mark the following statements as true (T) or false (F):

	T / F
If the current dictionary is degenerate, then the next pivot will not change the objective function.	F
Consider the version of the simplex method in which an artificial variable x_0 is introduced in phase-I. The phase-I problem can be unbounded.	F
After a degenerate pivot, the dictionary must still be degenerate.	T
An optimal solution to a linear programming problem must be a vertex of the set of feasible solutions.	F
No problem with an unbounded feasible region has an optimal solution.	F
Consider a dictionary. The primal and dual solutions associated with this dictionary must satisfy the complementarity relations: $x_j z_j = 0$ for all j and $w_i y_i = 0$ for all i.	T
It is possible to move from one vertex of the feasible set to any other vertex in a single pivot.	F
A dual pivot never decreases the value of the primal objective function.	F
A problem has an optimal solution if and only if it is both primal and dual feasible.	T

NOTE: *SAT-style grading scheme: 2 points for a correct answer, 0 points for an incorrect answer, and 1 point for no answer.*

(4) *(7 pts.)* The CEO of a large corporation needs the solution to the following linear programming problem:

$$\begin{array}{lrcrcrcr} \text{maximize} & -2x_1 & - & 2x_2 & - & 6x_3 & & \\ \text{subject to} & x_1 & - & 2x_2 & - & 2x_3 & \leq & -15 \\ & -2x_1 & + & x_2 & - & x_3 & \leq & -30 \\ & & & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Because the numbers represent millions of dollars, an error would be very costly. Hence, he's independently delegated the task of computing the solution to two subordinates: Alice and Brad. Alice came back with the following values for the primal and, by the way, for the associated dual values:

$$x_1^* = 25,\ x_2^* = 20,\ x_3^* = 0 \quad \text{and} \quad y_1^* = 2,\ y_2^* = 2$$

Brad provided the following values, which are different from Alice's:

$$x_1^* = 9,\ x_2^* = 0,\ x_3^* = 12$$

Since the two solutions disagree, you're being asked to determine who, if anyone, is correct. So, who is? Explain.

See next page.

(5) *(7 pts.)* Given n girls and m boys, you, the matchmaker, are to determine how to pairwise match the girls with the boys so as to maximize some overall desirability index. This problem can be set up as follows. Let x_{gb} be a decision variable that is 1 if girl g gets matched to boy b and is zero otherwise. Suppose that there is a desirability index d_{gb} associated with matching girl g with boy b. The problem is to maximize the overall desirability of all of the matches subject to the constraints that each girl gets matched with one boy and each boy gets matched with at most one girl. Formulate this problem as a linear programming problem. *Notes: (1) don't worry about specific data, just describe the problem; (2) you can write it either in AMPL or just in common mathematical notation; (3) don't worry that the solution might involve fractions.*

See page after next.

4. Check Alice's solution...

Primal feas: $x_1^* - 2x_2^* - 2x_3^* = 25 - 40 - 0 = -65 \leq -15$ ✓

$-2x_1^* + x_2^* - x_3^* = -50 + 20 - 0 = -30 \leq -30$ ✓

$x_1^*, x_2^*, x_3^* \geq 0$

Yes!

Dual feas.: $y_1^* - 2y_2^* = 2 - 4 = -2 \geq -2$ ✓

$-2y_1^* + y_2^* = -4 + 2 = -2 \geq -2$ ✓

$-2y_1^* - y_2^* = -4 - 2 = -6 \geq -6$ ✓

$y_1^*, y_2^*, y_3^* \geq 0$ Yes!

Duality gap: $-2x_1^* - 2x_2^* - 6x_3^* = -50 - 40 - 0 = -90$

$-15y_1^* - 30y_2^* = -30 - 60 = -90$

Yes!

Alice is optimal!

Check Brad's solution...

Agrees with Alice

Obj. value: $-2x_1^* - 2x_2^* - 6x_3^* = -18 - 72 = -90$

Primal Feas: $x_1^* - 2x_2^* - 2x_3^* = 9 - 24 = -15 \leq -15$ ✓

$-2x_1^* + x_2^* - x_3^* = -18 - 12 = -30 \leq -30$ ✓

Brad is also optimal!

5. $\max \sum_g \sum_b d_{gb} x_{gb}$

$$\sum_b x_{gb} = 1 \quad \text{for every girl } g$$

$$\sum_g x_{gb} \leq 1 \quad \text{for every boy } b$$

$$x_{gb} \geq 0 \quad \text{for every pair } (g,b)$$

(6) *(8 pts.)* The solution associated with the following dictionary is optimal:

$$\begin{array}{rcl} \zeta &=& 5 \qquad\qquad - \; 2w_2 \\ \hline x_1 &=& 6 + 5w_1 + \;\; w_2 \\ x_2 &=& 0 - \;\; w_1 - \; 7w_2 \end{array}$$

(a) Write down the optimal values for all of the variables in this problem.

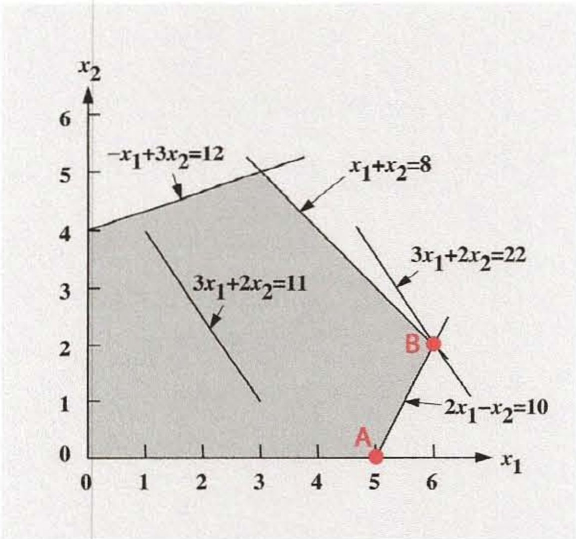
(b) If there are other optimal solutions, find one. If there are none, say how you know this to be the case.

See next page

(7) *(6 pts.)* Consider the following linear programming problem in 2 variables:

$$\begin{array}{lrcrcl} \text{maximize} & 3x_1 & + & 2x_2 & & \\ \text{subject to} & -x_1 & + & 3x_2 & \leq & 12 \\ & x_1 & + & x_2 & \leq & 8 \\ & 2x_1 & - & x_2 & \leq & 10 \\ & & & x_1, \, x_2 & \geq & 0. \end{array}$$

Here is a graph of the feasible region shown together with two level sets of the objective function:



Using the primal simplex method, one can pivot from the vertex labeled A to the vertex labeled B. Consider this pivot.

(a) What is the entering variable? x_2

(b) What is the leaving variable? w_2

Explain.

The nonbasic variables at A are: x_2 & w_3

The nonbasic variables at B are: w_2 & w_3

6. (a) $x_1 = 6$, $x_2 = 0$ (and $w_1 = 0$, $w_2 = 0$).

(b) The optimal dictionary has a nonbasic variable with a zero coefficient: $0w_1$.

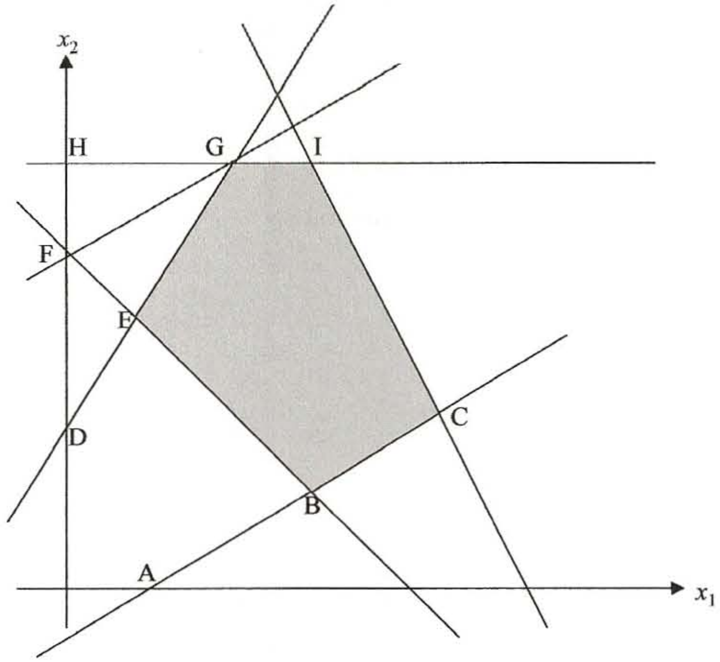
If we pick w_1 as an "entering" variable, perhaps we can do a non-degenerate pivot to a different, still optimal, solution. Let's try. Oh, oh. The dictionary is primal degenerate and the leaving variable is x_2 resulting in a degenerate pivot...

$$
\begin{aligned}
\zeta &= 5 + 0x_2 - 2w_2 \\
x_1 &= 6 - 5x_2 - 34w_2 \\
w_1 &= 0 - x_2 - 7w_2
\end{aligned}
\quad \Rightarrow \quad x_1 = 6,\ x_2 = 0
$$

same point!

Can we continue? Well, using the same idea, we could let x_2 enter and w_1 leave. But this brings us back to where we started. Hence, there is $\boxed{\text{only one}}$ optimal solution.

(8) *(5 pts.)* Consider a problem whose constraints are as shown here:



Suppose that the objective is to minimize x_1. Clearly, the optimal solution is at vertex E. It is not possible for the dual simplex method starting at vertex H to visit vertices G then E? Explain why. The dual simplex method is used when a dictionary solution is primal infeasible. As soon as primal feasibility is attained, we are optimal. Hence, we can't visit G before F because G is feasible, but not optimal.

(9) *(5 pts.)* Using the largest coefficient rule to choose the entering variable in the following primal feasible dictionary, compute the dictionary after **one iteration** of the simplex method.

$$
\begin{array}{rcl}
\zeta & = & \quad\;\; x_1 + 3x_2 - x_3 \\
\hline
w_1 & = & 1 + x_1 - 2x_2 - 6x_3 \\
\rightarrow\; w_2 & = & 0 \qquad\;\; - x_2 - 2x_3
\end{array}
$$

$$
\boxed{
\begin{array}{rcl}
\zeta & = & 0 + x_1 - 3w_2 - 7x_3 \\
\hline
w_1 & = & 1 + x_1 + 2w_2 - 2x_3 \\
x_2 & = & 0 \qquad\;\; - w_2 - 2x_3
\end{array}
}
$$