Princeton University
Department of Operations Research
and Financial Engineering

ORF 307
Optimization

Practice Midterm 2

Closed book. No computers, no calculators, no watches, no cellphones.

You may use a one-page two-sided cheat sheet.

Return the exam questions and your cheat sheet with your exam booklet.

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(1) *(28 pts.)* Consider the following problem:

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & & x_1,\ x_2 & \geq & 0 \end{array}$$

(a) Write the problem in dictionary form.
(b) Graph the feasible set (graph paper is attached–see last page).
(c) Solve this problem using the dual-phase-I/primal-phase-II simplex method (do each pivot by hand—there aren't many and the arithmetic is fairly simple).
(d) Plot and label the "solutions" associated with each dictionary on your graph.

(e) Write the dual problem as a minimization problem with greater-than constraints.
(f) Graph the feasible set for the dual problem.
(g) Plot the sequence of dual solutions on your graph of the dual.
(h) Assuming that the dual problem has an optimal solution, what are the optimal values of the dual variables and the dual slacks?

(2) *(12 pts.)* Consider the following problem:

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & 3x_2 & & \\ \text{subject to} & 5x_1 & - & 2x_2 & \leq & -1 \\ & 3x_1 & + & x_2 & = & 3 \\ & & & x_1,\ x_2 & \geq & 0 \end{array}$$

(a) Note that the second constraint is an *equality* constraint. Reexpress this problem in *standard form* (i.e., with less-than-or-equal-to constraints).

(b) Write down the dual of the standard-form problem from part (a).

(c) Assuming that the dual problem you wrote in part (b) has more than two dual variables, use the standard technique to reexpress the problem so that it has two dual variables.

(3) *(4 pts.)* Suppose you are solving an LP using the simplex method. The initial dictionary was feasible, so you are using just the primal simplex method. Primal feasibility was preserved until your most recent pivot which brought you to this primal infeasible dictionary:

maximize ζ	=	9	+	-7	w_1	+	11	x_2	+	23	x_3	+	5	w_3
x_1	=	1/2	-	1	w_1	-	5	x_2	-	-2	x_3	-	-1	w_3
w_2	=	1	-	-1	w_1	-	-4	x_2	-	0	x_3	-	0	w_3
x_4	=	1	-	0	w_1	-	1	x_2	-	1	x_3	-	4	w_3
w_4	=	-1/2	-	-3	w_1	-	-7	x_2	-	1	x_3	-	-3	w_3

$x_1,\ x_2,\ x_3,\ x_4,\ \ w_1,\ w_2,\ w_3,\ w_4\ \geq 0$

The pivot that brought you here had x_1 as the entering variable and w_1 as the leaving variable. What would have been a correct choice of entering and leaving variable?

(4) *(4 pts.)* Consider the following dual feasible dictionary:

ζ	=	0	+	-6	x_1	+	-7	x_2	+	-2	x_3
w_1	=	1	-	-1	x_1	-	2	x_2	-	-2	x_3
w_2	=	-4	-	-1	x_1	-	-2	x_2	-	-1	x_3
w_3	=	-1	-	2	x_1	-	-3	x_2	-	2	x_3
w_4	=	4	-	3	x_1	-	1	x_2	-	-1	x_3

Identify a valid leaving/entering variable pair for a pivot of the dual simplex method.

(5) *(4 pts.)* Consider the problem whose initial dictionary is this:

$$\begin{array}{rcrcrcrcr} \zeta & = & 1 & - & x_1 & + & x_2 & - & x_3 \\ \hline w_1 & = & 1 & - & x_1 & - & 2x_2 & + & 5x_3 \\ w_2 & = & 0 & & & - & x_2 & + & 2x_3 \end{array}$$

Solve this problem using the primal simplex method.

(6) *(12 pts.)* Suppose that you did some lab experiment and collected a bunch of numbers:

$$\begin{array}{c} 3.054 \\ 3.217 \\ 3.165 \\ 3.676 \\ 3.574 \\ 4.154 \\ 3.028 \\ 3.837 \\ \vdots \\ 3.035 \end{array}$$

Let's suppose that n is the number of numbers collected. In a paper you are going to submit for publication, you refer to these numbers as b_j, $j = 1, 2, \ldots, n$. In your paper, you explain that these numbers are measurements of a physical quantity β and that you used your data to estimate β by solving this optimization problem

$$\min_{\beta} \sum_{j=1}^{n} |\beta - b_j|.$$

(a) Show how to reformulate this problem as a linear programming problem.
(b) Write an AMPL model one can use to solve this linear programming problem. (Note: Just give the "model" – you can skip the "data" section.)

(7) *(18 pts.)* Mark the following statements as true (T) or false (F).

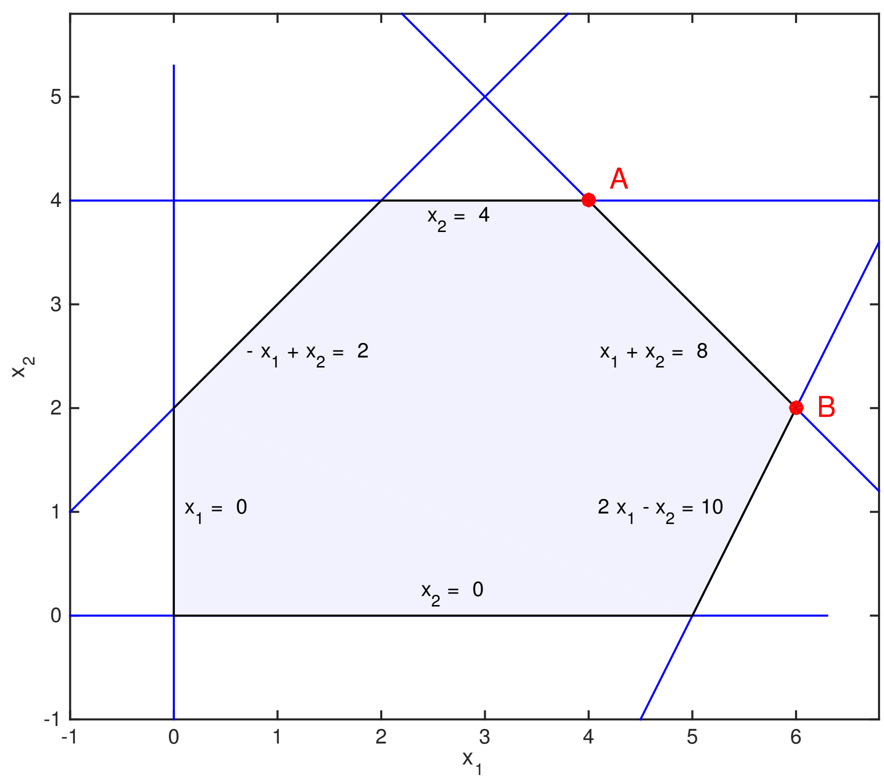
NOTE: We'll use an SAT-style grading scheme on this problem: 2 points for a correct answer, 0 points for an incorrect answer, and 1 point for no answer.

		T / F
(a)	If the feasible set extends to infinity in some directions, then the problem must be unbounded.	
(b)	The only way the Simplex method can fail is for it to cycle.	
(c)	In the Klee-Minty problem, if one replaces the largest-coefficient rule for the entering variable with *the variable with the smallest positive coefficient*, then the Simplex method finds the optimal solution in just one pivot.	
(d)	Using the perturbation method if necessary, every iteration of the primal simplex method increases the objective function.	
(e)	In a problem with n variables and m constraints, there are exactly $\binom{n+m}{n}$ dictionaries (some feasible, some not).	
(f)	Consider a problem with 2 variables (and any number of constraints). Assume that none of the vertices of the feasible polygon are degenerate. At every pivot after the first pivot, there can only be one unique choice for the entering variable (i.e., only one positive coefficient in the objective function).	
(g)	After a pivot of the simplex method, the coefficient in the objective function of the variable that just left the set of basic variables (and is now nonbasic) must be negative.	
(h)	Consider a linear programming problem with n nonnegative variables. If, in addition to being nonnegative, the only other constraints are that each of the n variables has an upper bound of 2, then no matter how one pivots, there can be no infeasible dictionaries ever encountered.	
(i)	If the current dictionary is degenerate, then the next pivot will not change the objective function value.	

(8) *(6 pts.)* Consider the following linear programming problem in 2 variables:

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & + & x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 2 \\ & & & x_2 & \leq & 4 \\ & x_1 & + & x_2 & \leq & 8 \\ & 2x_1 & - & x_2 & \leq & 10 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

Here is a graph of the feasible region:



Using the primal simplex method, one can pivot from the vertex labeled A to the vertex labeled B. Consider this pivot.

(a) What is the entering variable?

(b) What is the leaving variable?

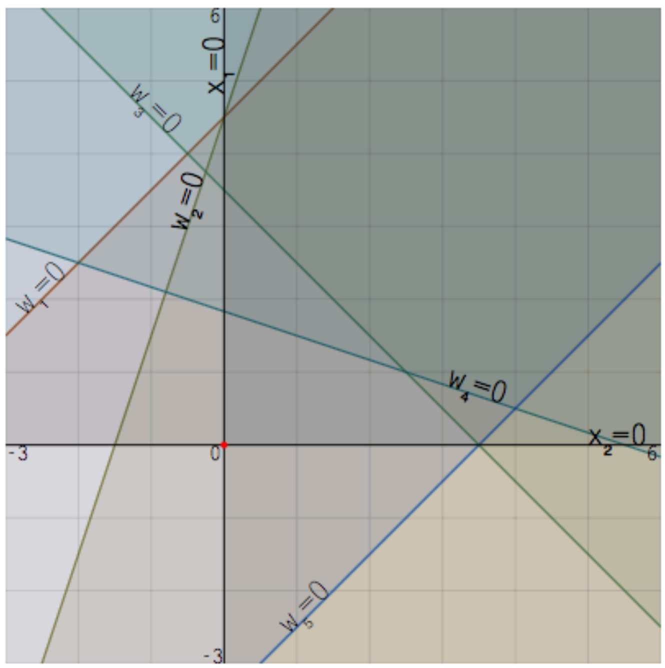
Explain.

(9) *(12 pts.)* Consider the following problem:

$$
\begin{aligned}
\text{maximize } \zeta &= 0 + 0\,x_1 + 1\,x_2 \\
w_1 &= 9 - (-2)\,x_1 - 2\,x_2 \\
w_2 &= 9 - (-6)\,x_1 - 2\,x_2 \\
w_3 &= -7 - (-2)\,x_1 - (-2)\,x_2 \\
w_4 &= -11 - (-2)\,x_1 - (-6)\,x_2 \\
w_5 &= 7 - 2\,x_1 - (-2)\,x_2
\end{aligned}
$$

$$x_1,\ x_2,\ \ w_1,\ w_2,\ w_3,\ w_4,\ w_5 \geq 0$$

Optimal   Infeasible   Unbounded



(a) If the problem has an optimal solution, identify the basic and nonbasic variables for the optimal dictionary. If, on the other hand, the problem is unbounded, identify the basic and nonbasic variables whose associated dictionary demonstrates the unboundedness.

(b) Identify a sequence of pivots (by stating pairs of entering/leaving variables) that can take us from the initial dictionary (in which x_1 and x_2 are the nonbasic variables) to an optimal or unbounded dictionary.

(c) We say that a constraint is *redundant* if removing it does not change the set of feasible points. Does this problem have any redundant constraints? If yes, identify them.