ORF 307: Lecture 9

Linear Programming: Chapter 6
Matrix Notation

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An Example

Consider

$$
\begin{array}{lrcrcrcl}
\text{maximize} & 3x_1 & + & 4x_2 & - & 2x_3 & & \\
\text{subject to} & x_1 & + & 0.5x_2 & - & 5x_3 & \leq & 2 \\
 & 2x_1 & - & x_2 & + & 3x_3 & \leq & 3 \\
 & & & x_1, & & x_2, \; x_3 & \geq & 0.
\end{array}
$$

Add slacks (using x's for slack variables):

$$
\begin{array}{rcrcrcrcrcl}
x_1 & + & 0.5x_2 & - & 5x_3 & + & x_4 & & & = & 2 \\
2x_1 & - & x_2 & + & 3x_3 & & & + & x_5 & = & 3.
\end{array}
$$

Cast constraints into matrix notation:

$$
\left[\begin{array}{rrr|rr} 1 & 0.5 & -5 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{array}\right]
\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline x_4 \\ x_5 \end{array}\right]
=
\left[\begin{array}{c} 2 \\ 3 \end{array}\right].
$$

Similarly cast objective function:

$$\left[\begin{array}{r} 3 \\ 4 \\ -2 \\ \hline 0 \\ 0 \end{array}\right]^T \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline {\color{blue}x_4} \\ {\color{blue}x_5} \end{array}\right].$$

In general, we have:

$$\boxed{\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & \quad x \geq 0. \end{array}}$$

Down the Road

Basic Variables: x_2, x_5.

Nonbasic Variables: x_1, x_3, x_4.

$$
\begin{aligned}
Ax &= \begin{bmatrix} 1 & 0.5 & -5 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \\
&= \begin{bmatrix} x_1 + 0.5x_2 - 5x_3 + x_4 \\ 2x_1 - x_2 + 3x_3 + x_5 \end{bmatrix} \\
&= \begin{bmatrix} 0.5x_2 + x_1 - 5x_3 + x_4 \\ -x_2 + x_5 + 2x_1 + 3x_3 \end{bmatrix} \\
&= \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \\
&= Bx_{\mathcal{B}} + Nx_{\mathcal{N}}.
\end{aligned}
$$

General Matrix Notation

Up to a rearrangement of columns,

$$A \overset{\mathrm{R}}{=} \left[\begin{array}{cc} B & N \end{array} \right]$$

Similarly, rearrange rows of x and c:

$$x \overset{\mathrm{R}}{=} \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix} \qquad c \overset{\mathrm{R}}{=} \begin{bmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{bmatrix}$$

Constraints:

$$Ax = b \quad \Longleftrightarrow \quad Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b$$

Objective:

$$\zeta = c^T x \quad \Longleftrightarrow \quad c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}}$$

Matrix B is $m \times m$ and invertible! Why?

Express $x_{\mathcal{B}}$ and ζ in terms of $x_{\mathcal{N}}$:

$$\begin{aligned} x_{\mathcal{B}} &= B^{-1}b - B^{-1}Nx_{\mathcal{N}} \\ \zeta &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1}b - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}}. \end{aligned}$$

Dictionary in Matrix Notation

$$\begin{array}{rcccl} \zeta &=& c_{\mathcal{B}}^T B^{-1}b &-& \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &=& B^{-1}b &-& B^{-1}Nx_{\mathcal{N}}. \end{array}$$

Example Revisited

$$B = \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \quad \Longrightarrow \quad B^{-1} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -10 & 2 \\ 4 & -7 & 2 \end{bmatrix}$$

$$(B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} = \begin{bmatrix} 2 & 4 \\ -10 & -7 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -38 \\ 8 \end{bmatrix}$$

$$c_{\mathcal{B}}^T B^{-1} b = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 16$$

Sanity Check

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & 3x_1 & + & 4x_2 & - & 2x_3 \\
\hline
{\color{blue}x_4} & = & 2 & - & x_1 & - & 0.5x_2 & + & 5x_3 \\
{\color{blue}x_5} & = & 3 & - & 2x_1 & + & x_2 & - & 3x_3.
\end{array}
$$

Let x_2 enter and ${\color{blue}x_4}$ leave.

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 16 & - & 5x_1 & - & 8{\color{blue}x_4} & + & 38x_3 \\
\hline
x_2 & = & 4 & - & 2x_1 & - & 2{\color{blue}x_4} & + & 10x_3 \\
{\color{blue}x_5} & = & 7 & - & 4x_1 & - & 2{\color{blue}x_4} & + & 7x_3.
\end{array}
$$

Dual Stuff

Associated Primal Solution:

$$\begin{aligned} x^*_{\mathcal{N}} &= 0 \\ x^*_{\mathcal{B}} &= B^{-1}b \end{aligned}$$

Dual Variables:

$$\begin{aligned} (x_1, \ldots, x_n, w_1, \ldots, w_m) &\longrightarrow (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \\ (z_1, \ldots, z_n, y_1, \ldots, y_m) &\longrightarrow (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) \end{aligned}$$

Associated Dual Solution:

$$\begin{aligned} z^*_{\mathcal{B}} &= 0 \\ z^*_{\mathcal{N}} &= (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \end{aligned}$$

Associated Solution Value:

$$\zeta^* = c_{\mathcal{B}}^T B^{-1} b$$

Primal Dictionary:

$$\boxed{\begin{aligned} \zeta &= \zeta^* - {z^*_{\mathcal{N}}}^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x^*_{\mathcal{B}} - B^{-1} N x_{\mathcal{N}}. \end{aligned}}$$

Dual Dictionary:

$$\boxed{\begin{aligned} -\xi &= -\zeta^* - {x^*_{\mathcal{B}}}^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z^*_{\mathcal{N}} + (B^{-1}N)^T z_{\mathcal{B}}. \end{aligned}}$$

Dual of Problems in “Equality” Form

Consider:

$$\begin{aligned} \max c^T x & \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Rewrite equality constraints as pairs of in-equalities:

$$\begin{aligned} \max c^T x & \\ Ax &\leq b \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Put into block-matrix form:

$$\begin{aligned} \max c^T x & \\ \begin{bmatrix} A \\ -A \end{bmatrix} x &\begin{matrix} \leq \\ \leq \end{matrix} \begin{bmatrix} b \\ -b \end{bmatrix} \\ x &\geq 0 \end{aligned}$$

Dual of Problems in "Equality" Form

Consider:

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Put into block-matrix form:

$$\begin{aligned} \max c^T x & \\ \begin{bmatrix} A \\ -A \end{bmatrix} x &\begin{matrix} \leq \\ \leq \end{matrix} \begin{bmatrix} b \\ -b \end{bmatrix} \\ x &\geq 0 \end{aligned}$$

Dual is:

$$\begin{aligned} \min \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & \\ \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} &\geq c \\ y^+, y^- &\geq 0 \end{aligned}$$

Which is equivalent to:

$$\begin{aligned} \min b^T (y^+ - y^-) & \\ A^T (y^+ - y^-) &\geq c \\ y^+, y^- &\geq 0 \end{aligned}$$

Finally, letting $y = y^+ - y^-$, we get

$$\begin{aligned} \min b^T y & \\ A^T y &\geq c \\ y & \quad \text{free.} \end{aligned}$$

Dual of Problems in General Form

- Equality constraints $\Longrightarrow$ free variables in dual.
- Inequality constraints $\Longrightarrow$ nonnegative variables in dual.

Corollary:

- Free variables $\Longrightarrow$ equality constraints in dual.
- Nonnegative variables $\Longrightarrow$ inequality constraints in dual.