ORF 307: Lecture 7

Linear Programming: Chapter 5
Duality II

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Feb 26, 2019

Slides last edited on February 28, 2019

ORFEntertainment
Thursday
February 28th
7:30 PM
Sherrerd Hall Atrium

ALL ARE WELCOME, PRIZES, FOOD, FUN

PERFORMANCE TIMES ARE STILL AVAILABLE.... IT'S NOT TOO LATE TO SIGN UP!!!

NEW THIS YEAR KARAOKE!!!

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Complementary Slackness

Primal Problem:

$$\begin{array}{lll} \max & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j + w_i = b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \\ & w_i \geq 0 & i = 1, \ldots, m \end{array}$$

Dual Problem:

$$\begin{array}{lll} \min & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} - z_j = c_j & j = 1, \ldots, n \\ & y_i \geq 0 & i = 1, \ldots, m \\ & z_j \geq 0 & j = 1, \ldots, n \end{array}$$

Theorem. *At optimality, we have*

$$\begin{array}{rcll} x_j z_j & = & 0, & \textit{for } j = 1, 2, \ldots, n, \\ w_i y_i & = & 0, & \textit{for } i = 1, 2, \ldots, m. \end{array}$$

Proof

Recall the proof of the Weak Duality Theorem:

$$\begin{aligned}\sum_j c_j x_j &\leq \sum_j (c_j + z_j) x_j = \sum_j \left(\sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j \\ &= \sum_i \left(\sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \leq \sum_i b_i y_i,\end{aligned}$$

The inequalities come from the fact that

$$\begin{aligned} x_j z_j &\geq 0, \quad \text{for all } j, \\ w_i y_i &\geq 0, \quad \text{for all } i. \end{aligned}$$

By Strong Duality Theorem, the inequalities are equalities at optimality.

Dual Simplex Method

Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

An Example. Showing both primal and dual dictionaries:

maximize ζ	=	0	+	-11	x_1	+	-3	x_2	+	-2	x_3	+	-6	x_4
w_1	=	0	-	3	x_1	-	1	x_2	-	1	x_3	-	-3	x_4
w_2	=	-2	-	-1	x_1	-	3	x_2	-	-1	x_3	-	0	x_4
w_3	=	-4	-	-2	x_1	-	-2	x_2	-	6	x_3	-	1	x_4

maximize -ξ	=	0	+	0	y_1	+	2	y_2	+	4	y_3
z_1	=	11	-	-3	y_1	-	1	y_2	-	2	y_3
z_2	=	3	-	-1	y_1	-	-3	y_2	-	2	y_3
z_3	=	2	-	-1	y_1	-	1	y_2	-	-6	y_3
z_4	=	6	-	3	y_1	-	0	y_2	-	-1	y_3

Looking at dual dictionary: y_3 enters, z_2 leaves.

On the primal dictionary: w_3 leaves, x_2 enters.

After pivot...

(Seed = 3, generate 3x, with negation)

Dual Simplex Method: Second Pivot

Going in, we have:

$$
\begin{array}{lcrcrlcrlcrlcrl}
\text{maximize } \zeta & = & -6 & + & -8 & x_1 & + & -3/2 & w_3 & + & -11 & x_3 & + & -15/2 & x_4 \\
w_1 & = & -2 & - & 2 & x_1 & - & 1/2 & w_3 & - & 4 & x_3 & - & -5/2 & x_4 \\
w_2 & = & -8 & - & -4 & x_1 & - & 3/2 & w_3 & - & 8 & x_3 & - & 3/2 & x_4 \\
x_2 & = & 2 & - & 1 & x_1 & - & -1/2 & w_3 & - & -3 & x_3 & - & -1/2 & x_4
\end{array}
$$

$$
\begin{array}{lcrcrlcrlcrl}
\text{maximize } -\xi & = & 6 & + & 2 & y_1 & + & 8 & y_2 & + & -2 & z_2 \\
z_1 & = & 8 & - & -2 & y_1 & - & 4 & y_2 & - & -1 & z_2 \\
y_3 & = & 3/2 & - & -1/2 & y_1 & - & -3/2 & y_2 & - & 1/2 & z_2 \\
z_3 & = & 11 & - & -4 & y_1 & - & -8 & y_2 & - & 3 & z_2 \\
z_4 & = & 15/2 & - & 5/2 & y_1 & - & -3/2 & y_2 & - & 1/2 & z_2
\end{array}
$$

Looking at dual: y_2 enters, z_1 leaves.

Looking at primal: w_2 leaves, x_1 enters.

Dual Simplex Method Pivot Rule

$$\text{maximize } \zeta = -6 + (-8)\,x_1 + (-3/2)\,w_3 + (-11)\,x_3 + (-15/2)\,x_4$$

$$\begin{aligned}
w_1 &= \boxed{-2} - 2\,x_1 - 1/2\,w_3 - 4\,x_3 - (-5/2)\,x_4 \\
w_2 &= \boxed{-8} - (-4)\,x_1 - 3/2\,w_3 - 8\,x_3 - 3/2\,x_4 \\
x_2 &= 2 - 1\,x_1 - (-1/2)\,w_3 - (-3)\,x_3 - (-1/2)\,x_4
\end{aligned}$$

Referring to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...

Dual Simplex Method: Third Pivot

Going in, we have:

$$
\begin{array}{lcrcrlcrlcrlcrl}
\text{maximize } \zeta & = & -22 & + & -2 & w_2 & + & -9/2 & w_3 & + & -27 & x_3 & + & -21/2 & x_4 \\
w_1 & = & -6 & - & 1/2 & w_2 & - & 5/4 & w_3 & - & 8 & x_3 & - & -7/4 & x_4 \\
x_1 & = & 2 & - & -1/4 & w_2 & - & -3/8 & w_3 & - & -2 & x_3 & - & -3/8 & x_4 \\
x_2 & = & 0 & - & 1/4 & w_2 & - & -1/8 & w_3 & - & -1 & x_3 & - & -1/8 & x_4
\end{array}
$$

Which variable must leave and which must enter?

See next page...

Dual Simplex Method: Third Pivot—Answer

Answer is: w_1 leaves, x_4 enters.

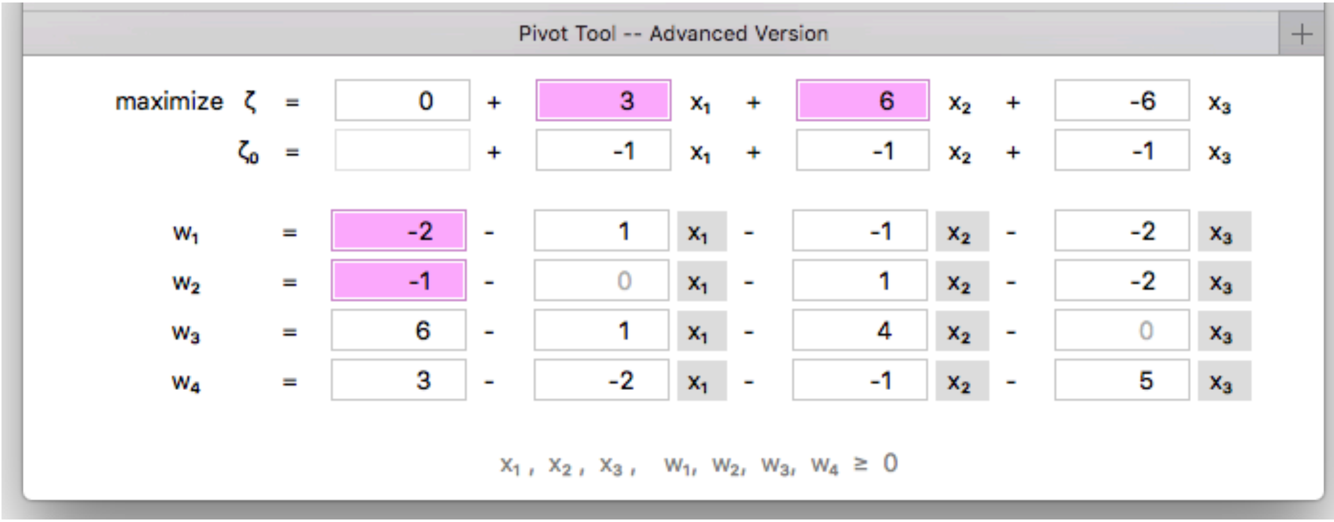
Resulting dictionary is OPTIMAL:

maximize ζ	=	-58	+	-5	w_2	+	-12	w_3	+	-75	x_3	+	-6	w_1
x_4	=	24/7	-	-2/7	w_2	-	-5/7	w_3	-	-32/7	x_3	-	-4/7	w_1
x_1	=	23/7	-	-5/14	w_2	-	-9/14	w_3	-	-26/7	x_3	-	-3/14	w_1
x_2	=	3/7	-	3/14	w_2	-	-3/14	w_3	-	-11/7	x_3	-	-1/14	w_1

Dual-Based Phase I Method

Dual-Based Phase I Method

Example:



Seed $= 4$

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it's *dual feasible*.

Phase I—First Pivot: w_1 leaves, x_3 enters.

Let's go pivoting...

Recall initial dictionary:

Pivot Tool -- Advanced Version

$$
\begin{array}{rcrcrcrcrc}
\text{maximize } \zeta & = & 0 & + & 3 & x_1 & + & 6 & x_2 & + & -6 & x_3 \\
\zeta_0 & = & & + & -1 & x_1 & + & -1 & x_2 & + & -1 & x_3 \\
w_1 & = & -2 & - & 1 & x_1 & - & -1 & x_2 & - & -2 & x_3 \\
w_2 & = & -1 & - & 0 & x_1 & - & 1 & x_2 & - & -2 & x_3 \\
w_3 & = & 6 & - & 1 & x_1 & - & 4 & x_2 & - & 0 & x_3 \\
w_4 & = & 3 & - & -2 & x_1 & - & -1 & x_2 & - & 5 & x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$$

Dual pivot: w_1 leaves, x_3 enters.

After pivot:

Pivot Tool -- Advanced Version

$$
\begin{array}{rcrcrcrcrc}
\text{maximize } \zeta & = & -6 & + & 0 & x_1 & + & 9 & x_2 & + & -3 & w_1 \\
\zeta_0 & = & & + & -3/2 & x_1 & + & -1/2 & x_2 & + & -1/2 & w_1 \\
x_3 & = & 1 & - & -1/2 & x_1 & - & 1/2 & x_2 & - & -1/2 & w_1 \\
w_2 & = & 1 & - & -1 & x_1 & - & 2 & x_2 & - & -1 & w_1 \\
w_3 & = & 6 & - & 1 & x_1 & - & 4 & x_2 & - & 0 & w_1 \\
w_4 & = & -2 & - & 1/2 & x_1 & - & -7/2 & x_2 & - & 5/2 & w_1
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$$

Recall current dictionary:

Pivot Tool -- Advanced Version

$$
\begin{array}{rcl}
\text{maximize} \quad \zeta & = & -6 + 0\,x_1 + 9\,x_2 + (-3)\,w_1 \\
\zeta_0 & = & + (-3/2)\,x_1 + (-1/2)\,x_2 + (-1/2)\,w_1 \\
x_3 & = & 1 - (-1/2)\,x_1 - 1/2\,x_2 - (-1/2)\,w_1 \\
w_2 & = & 1 - (-1)\,x_1 - 2\,x_2 - (-1)\,w_1 \\
w_3 & = & 6 - 1\,x_1 - 4\,x_2 - 0\,w_1 \\
w_4 & = & -2 - 1/2\,x_1 - (-7/2)\,x_2 - 5/2\,w_1
\end{array}
$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4 \geq 0$$

Dual pivot: w_4 leaves, x_2 enters.

After pivot:

Pivot Tool -- Advanced Version

$$
\begin{array}{rcl}
\text{maximize} \quad \zeta & = & -6/7 + 9/7\,x_1 + 18/7\,w_4 + 24/7\,w_1 \\
\zeta_0 & = & + (-11/7)\,x_1 + (-1/7)\,w_4 + (-6/7)\,w_1 \\
x_3 & = & 5/7 - (-3/7)\,x_1 - 1/7\,w_4 - (-1/7)\,w_1 \\
w_2 & = & -1/7 - (-5/7)\,x_1 - 4/7\,w_4 - 3/7\,w_1 \\
w_3 & = & 26/7 - 11/7\,x_1 - 8/7\,w_4 - 20/7\,w_1 \\
x_2 & = & 4/7 - (-1/7)\,x_1 - (-2/7)\,w_4 - (-5/7)\,w_1
\end{array}
$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4 \geq 0$$

Recall current dictionary:

Pivot Tool -- Advanced Version

maximize ζ	=	-6/7	+	9/7	x_1	+	18/7	w_4	+	24/7	w_1
ζ_0	=		+	-11/7	x_1	+	-1/7	w_4	+	-6/7	w_1
x_3	=	5/7	-	-3/7	x_1	-	1/7	w_4	-	-1/7	w_1
w_2	=	-1/7	-	-5/7	x_1	-	4/7	w_4	-	3/7	w_1
w_3	=	26/7	-	11/7	x_1	-	8/7	w_4	-	20/7	w_1
x_2	=	4/7	-	-1/7	x_1	-	-2/7	w_4	-	-5/7	w_1

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Dual pivot: w_2 leaves, x_1 enters.

After pivot:

Feasible!

Pivot Tool -- Advanced Version

maximize ζ	=	-3/5	+	9/5	w_2	+	18/5	w_4	+	21/5	w_1
ζ_0	=		+	-11/5	w_2	+	-7/5	w_4	+	-9/5	w_1
x_3	=	4/5	-	-3/5	w_2	-	-1/5	w_4	-	-2/5	w_1
x_1	=	1/5	-	-7/5	w_2	-	-4/5	w_4	-	-3/5	w_1
w_3	=	17/5	-	11/5	w_2	-	12/5	w_4	-	19/5	w_1
x_2	=	3/5	-	-1/5	w_2	-	-2/5	w_4	-	-4/5	w_1

$x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0$

Current dictionary is feasible:

Pivot Tool -- Advanced Version

maximize ζ	=	-3/5	+	9/5	w_2	+	18/5	w_4	+	21/5	w_1
ζ_0	=		+	-11/5	w_2	+	-7/5	w_4	+	-9/5	w_1
x_3	=	4/5	-	-3/5	w_2	-	-1/5	w_4	-	-2/5	w_1
x_1	=	1/5	-	-7/5	w_2	-	-4/5	w_4	-	-3/5	w_1
w_3	=	17/5	-	11/5	w_2	-	12/5	w_4	-	19/5	w_1
x_2	=	3/5	-	-1/5	w_2	-	-2/5	w_4	-	-4/5	w_1

$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Ignore fake objective. Use the real objective. Primal pivot: w_1 enters, w_3 leaves.

After pivot:

Pivot Tool -- Advanced Version

maximize ζ	=	60/19	+	-12/19	w_2	+	18/19	w_4	+	-21/19	w_3
ζ_0	=		+	-22/19	w_2	+	-5/19	w_4	+	9/19	w_3
x_3	=	22/19	-	-7/19	w_2	-	1/19	w_4	-	2/19	w_3
x_1	=	14/19	-	-20/19	w_2	-	-8/19	w_4	-	3/19	w_3
w_1	=	17/19	-	11/19	w_2	-	12/19	w_4	-	5/19	w_3
x_2	=	25/19	-	5/19	w_2	-	2/19	w_4	-	4/19	w_3

$x_1,\ x_2,\ x_3,\ \ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Getting close:

Pivot Tool -- Advanced Version

maximize ζ	=	60/19	+	-12/19	w_2	+	18/19	w_4	+	-21/19	w_3
ζ_0	=		+	-22/19	w_2	+	-5/19	w_4	+	9/19	w_3
x_3	=	22/19	-	-7/19	w_2	-	1/19	w_4	-	2/19	w_3
x_1	=	14/19	-	-20/19	w_2	-	-8/19	w_4	-	3/19	w_3
w_1	=	17/19	-	11/19	w_2	-	12/19	w_4	-	5/19	w_3
x_2	=	25/19	-	5/19	w_2	-	2/19	w_4	-	4/19	w_3

$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4 \geq 0$

Primal pivot: w_4 enters, w_1 leaves.

After pivot:

Optimal!

Pivot Tool -- Advanced Version

maximize ζ	=	9/2	+	-3/2	w_2	+	-3/2	w_1	+	-3/2	w_3
ζ_0	=		+	-11/12	w_2	+	5/12	w_1	+	7/12	w_3
x_3	=	13/12	-	-5/12	w_2	-	-1/12	w_1	-	1/12	w_3
x_1	=	4/3	-	-2/3	w_2	-	2/3	w_1	-	1/3	w_3
w_4	=	17/12	-	11/12	w_2	-	19/12	w_1	-	5/12	w_3
x_2	=	7/6	-	1/6	w_2	-	-1/6	w_1	-	1/6	w_3

$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4 \geq 0$