ORF 307: Lecture 13

Linear Programming: Chapter 14: Network Flows: Theory

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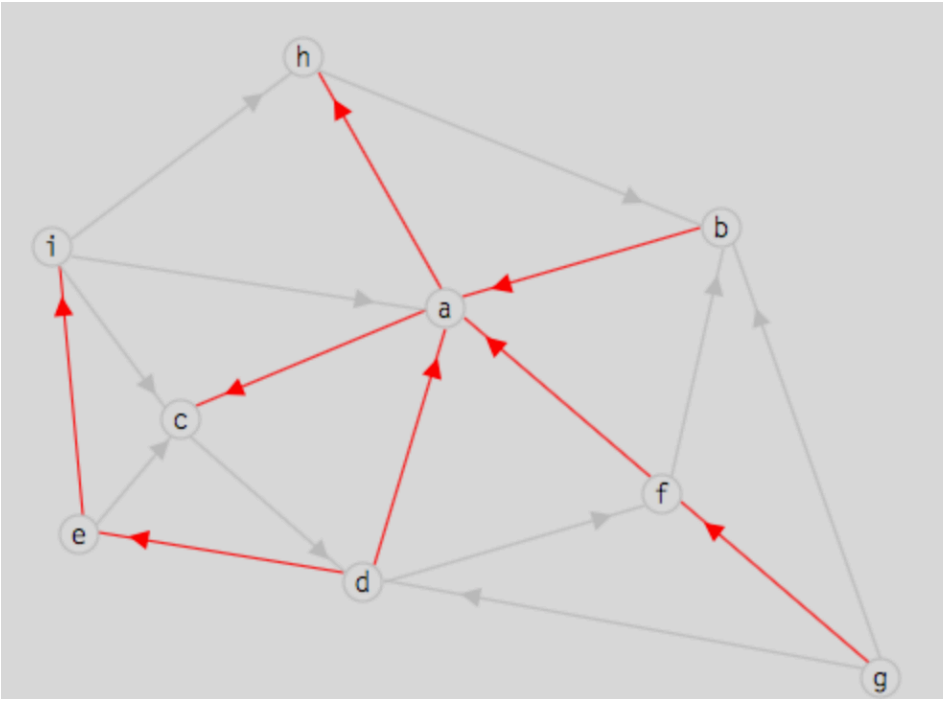
Networks

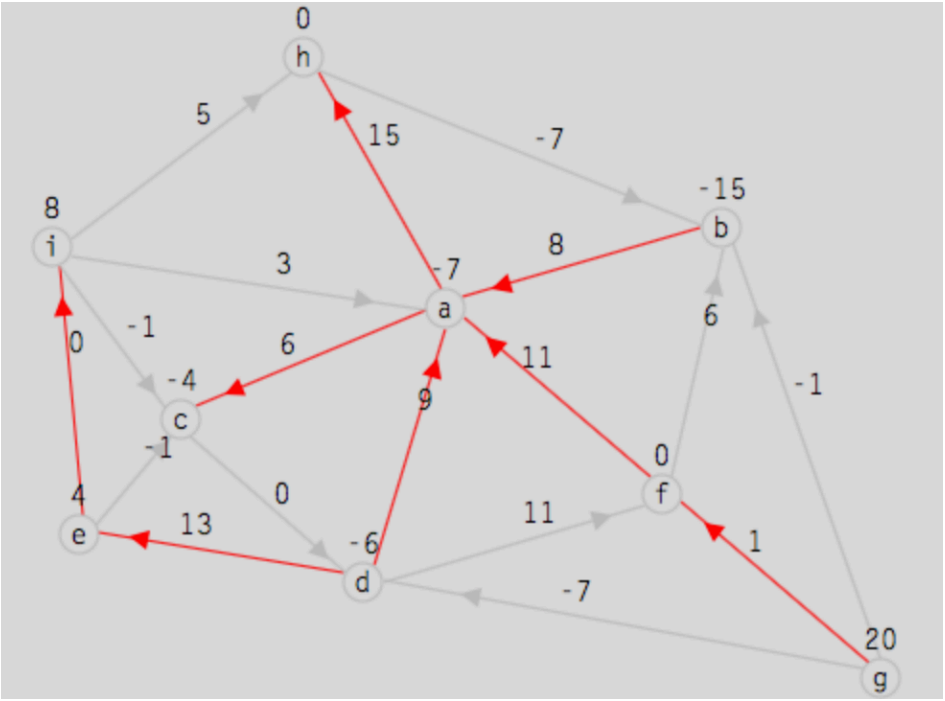


Structural elements:

- $\mathcal{N}$ *Nodes* (let m denote number of them).
- $\mathcal{A}$ Directed *Arcs*
  - subset of all possible arcs: $\{(i,j) : \ i,j \in \mathcal{N}, i \neq j\}$.
  - arcs are *directed*: $(i,j) \neq (j,i)$.

Note: seed $= 1111$, nodes $= 9$

Network Flow Data



Data:

- $b_i,\ i \in \mathcal{N}$, *supply* at node i
- $c_{ij},\ (i,j) \in \mathcal{A}$, *cost* of shipping 1 unit along arc (i,j).

Note: *demands* are recorded as *negative supplies*.

Note 2: Supplies/costs are shown *above* their nodes/arcs.

Network Flow Problem

Decision Variables:

x_{ij}, *quantity* to ship along arc (i, j).

Objective:

$$\text{minimize} \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

Constraints:

- "Mass" conservation (aka flow balance):

$$\text{inflow}(k) - \text{outflow}(k) = \text{demand}(k) = -\text{supply}(k), \qquad k \in \mathcal{N}$$

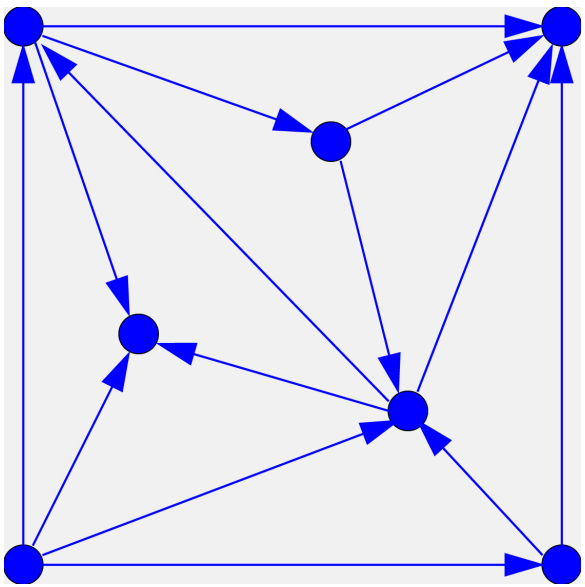
$$\Updownarrow$$

$$\sum_{\substack{i: \\ (i,k)\in\mathcal{A}}} x_{ik} - \sum_{\substack{j: \\ (k,j)\in\mathcal{A}}} x_{kj} = -b_k, \qquad k \in \mathcal{N}$$

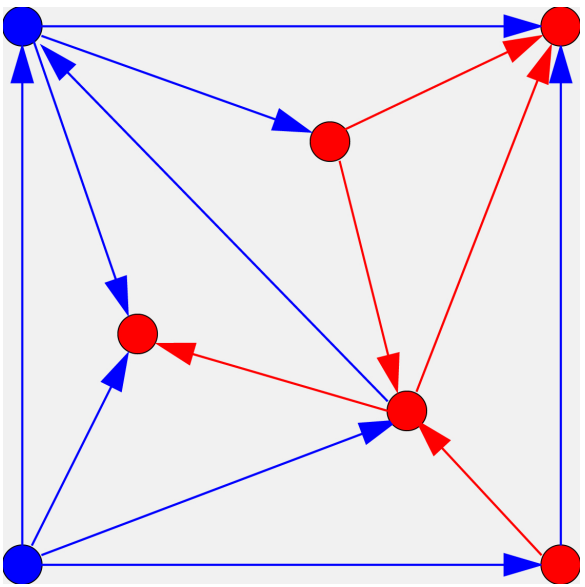
- Nonnegativity:

$$x_{ij} \geq 0, \qquad (i, j) \in \mathcal{A}$$

Definition: Subnetwork

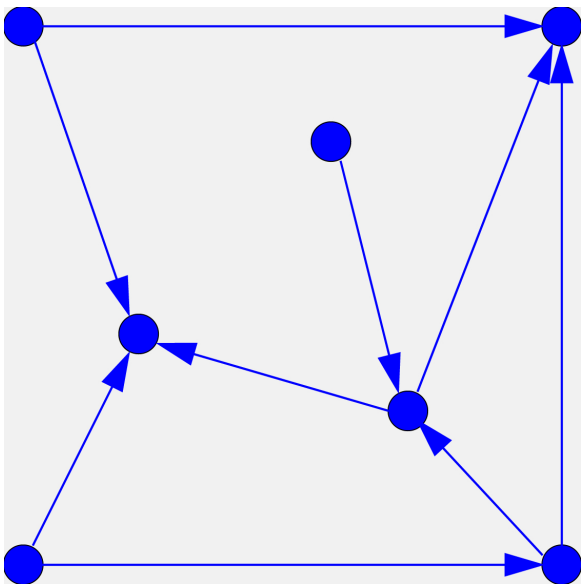


Network

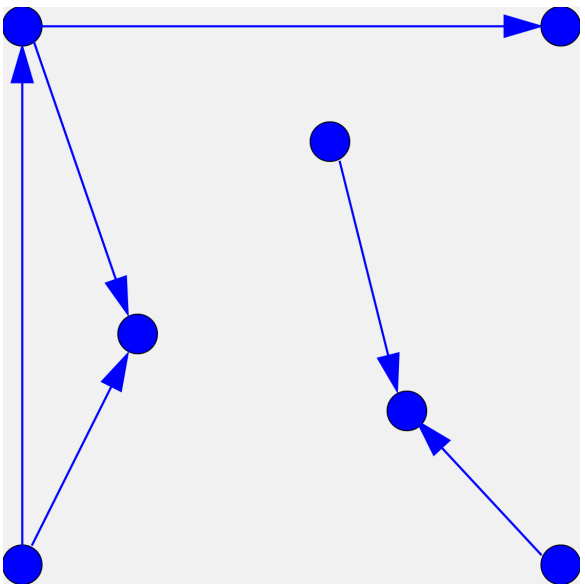


Subnetwork

Connected vs. Disconnected

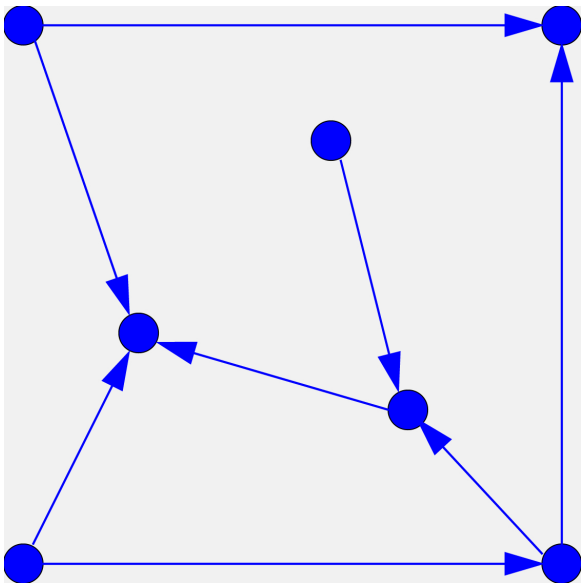


Connected

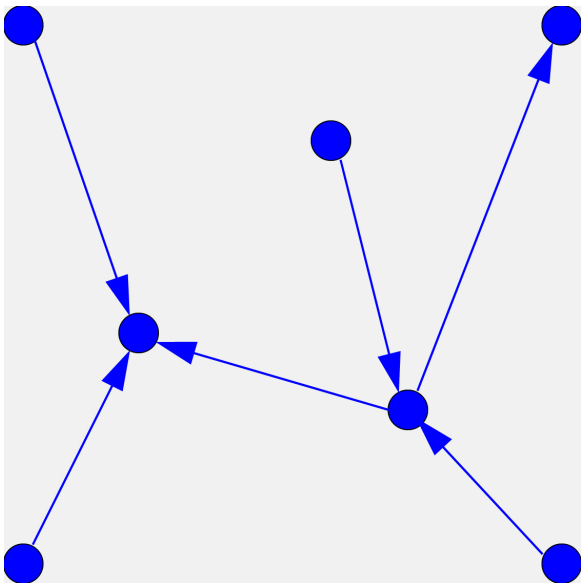


Disconnected

Cyclic vs. Acyclic

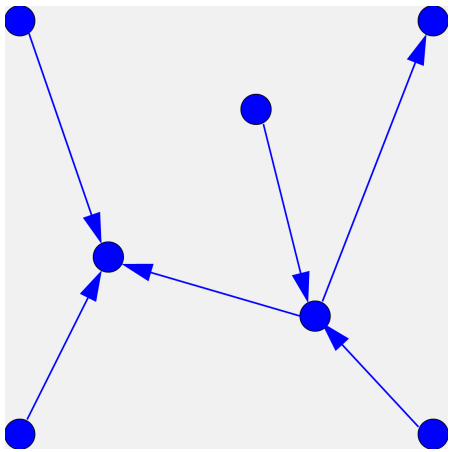


Cyclic

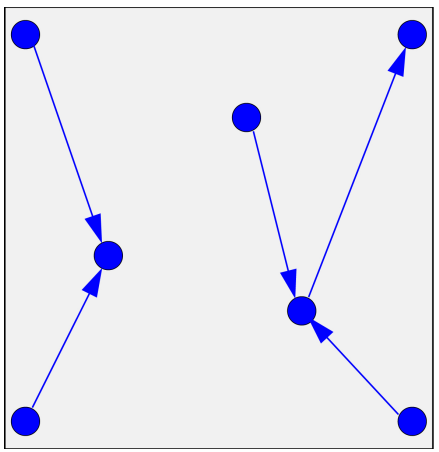


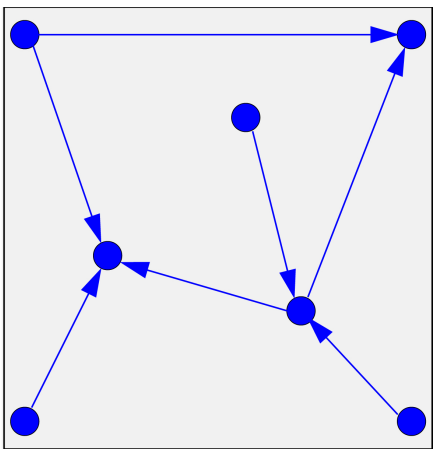
Acyclic

Trees



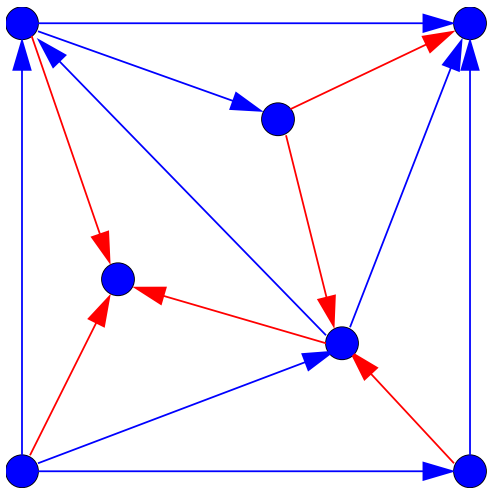
Tree = Connected + Acyclic





Not Trees

Spanning Trees



Spanning Tree – A tree touching every node

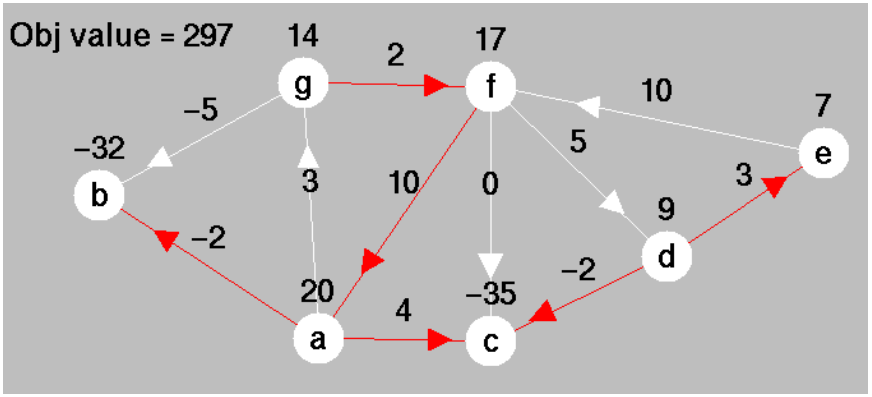
(No need to recolor the nodes.)

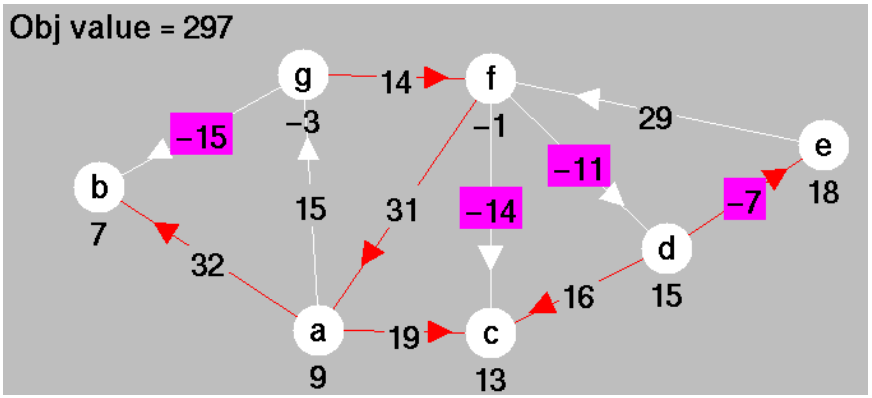
Tree Solution

$$x_{ij} = 0 \qquad \text{for } (i,j) \notin \text{Tree Arcs}$$

Note: Tree solutions are easy to compute—start at the "leaves" and work inward...

Online Pivot Tool–Notations





Data:

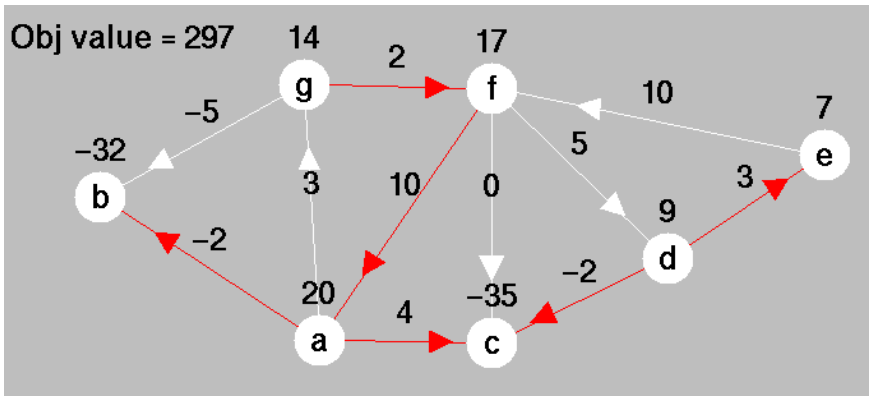
- *Costs* on arcs shown above arcs.
- *Supplies* at nodes shown above nodes.

Variables:

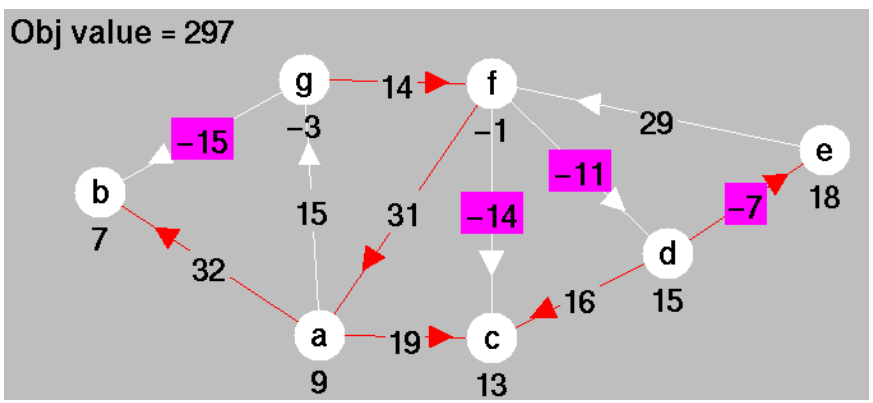
- *Primal flows* shown on tree arcs.
- *Dual slacks* shown on nontree arcs.
- *Dual variables* shown below nodes.

Tree Solutions–An Example

Data:



Variables:



- Fix a root node, say a.
- *Primal flows* on tree arcs calculated recursively from leaves inward.
- *Dual variables* at nodes calculated recursively from root node outward along tree arcs using:
  $y_j - y_i = c_{ij}$
- *Dual slacks* on nontree arcs calculated using:
  $z_{ij} = y_i - y_j + c_{ij}$.