ORF 245 Fundamentals of Statistics
Chapter 5
Probability

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Sample Spaces (aka Populations) and Events

When considering experiments *to be performed*, we need to define the *set of possible outcomes*.

The set is called the *sample space* and is usually denoted by Ω.

A sample space can be either finite or infinite.

Examples:

- Toss a coin: $\Omega = \{H, T\}$.
- Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Roll a pair of dice: $\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.
- How long from now until the next major earthquake in California: $\Omega = \{t \mid t \geq 0\}$.

Subsets of Ω are called *events*. They are usually denoted by capital letters like A or B.

Example:

- Roll a pair of dice and consider the event that the sum is 4:

$$A = \{(1, 3), (2, 2), (3, 1)\} \subset \Omega.$$

And, Or, Not, etc.

Intersection:	Event A *and* B both occur is written $A \cap B$.
Union:	Event A *or* B occurs is written $A \cup B$.
Complement:	Event A does *not* occur is written A^c.
Empty Set:	The set containing *no elements* is denoted $\emptyset$.
Disjoint:	Sets A and B are *disjoint* if $A \cap B = \emptyset$.

Probability

Probabilities are numbers assigned to events. They must satisfy the following properties:

- $P(\Omega) = 1$.
- $P(A) \geq 0$ for all $A \subset \Omega$.
- If A_1 and A_2 are disjoint, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

It follows that

- $P(A^c) = 1 - P(A)$.
- $P(\emptyset) = 0$.
- $P(A) \leq P(B)$ whenever $A \subset B$.

Finite Sample Spaces

Examples:

1. Flip two pennies and let $\Omega = \{(h,h),(h,t),(t,h),(t,t)\}$. Then,

$$\begin{aligned} P(\text{one head and one tail}) &= P(\{(h,t)\} \cup \{(t,h)\}) = P(\{(h,t)\}) + P(\{(t,h)\}) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \end{aligned}$$

2. Flip two pennies and let $\Omega = \{$ "two heads", "one head", "no heads" $\}$. Then,

$$P(\text{one head and one tail}) = \text{hmmm... hard to say}$$

3. Roll a pair of dice and let $\Omega = \{(i,j) \mid 1 \le i \le 6, 1 \le j \le 6\}$. Then,

$$P(\text{sum is } 4) = P(1,3) + P(2,2) + P(3,1) = 3/36$$

Conditional Probability

Definition. The conditional probability of A given that B is known to have occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Law of Total Probability. Let $B_1, B_2, \ldots, B_n$ be a disjoint collection of sets each having positive probability whose union is all of Ω. Then,

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Bayes' Rule. If, in addition to the assumptions above, $P(A) > 0$, then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}.$$

If the disjoint collection consists of just two sets, B and B^c, then the formula can be written more simply as

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Example of Bayes' Rule

Consider women of a certain given age and overall health status.

Facts:

- Probability that a woman has breast cancer $= P(B) = 1\%$.
- Probability that a mammogram will give a positive result (indicating cancer is present) (event A) for women who are known to have cancer $= P(A|B) = 80\%$.
- A women who is known not to have cancer will test positive 10% of the time. That is, $P(A|B^c) = 10\%$.

Question: If a woman tests positive for breast cancer, what is the probability that she actually has breast cancer? That is, what is $P(B|A)$?

Cancer doctors were asked this question. Most estimated the answer to be 75%.

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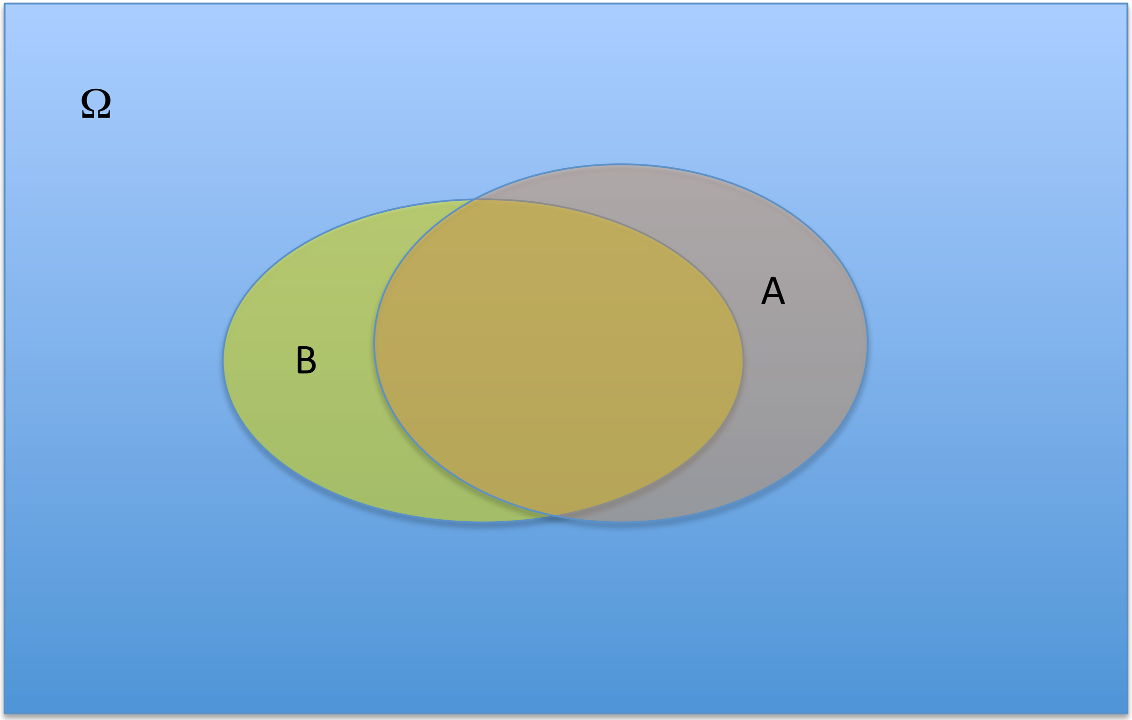
Let's compute:

$$P(B|A) = \frac{(0.8)(0.01)}{(0.8)(0.01) + (0.1)(0.99)} = \frac{8}{8+99} \approx 7.5\%.$$

Venn Diagram



Independence

Events A and B are independent means

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$$P(A \cap B) = P(A)P(B)$$

Example: Poker

In a game of poker, what is the probability that a five-card hand will contain

(a) a straight (five cards in a numerical sequence not all from same suit),

(b) four of a kind (four cards of one value), and

(c) a full house (three cards of one value and two cards of another)?

(a)

$$\frac{10 \cdot (4^5 - 4)}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} = 0.00392$$

(b)

$$\frac{13 \cdot 48}{\binom{52}{5}} = 0.000240$$

(c)

$$\frac{13 \cdot 12 \cdot \binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = 0.001441$$

Example: Birthday Problem

In a classroom of n students, what's the probability p_n that two (or more) students share the same birthday?

It's easier to compute the probability that no two students share a birthday.
Let's look at the students one at a time.
The first student can have any birthday he/she likes.
The second student cannont share the first student's birthday: 364 choices.
The third student cannot share either of the first two birthdays: 363 choices.
... Etc. ...
The n-th student cannot share any of the previous $n-1$ birthdays: $365-n+1$ choices.
Therefore, probability of no shared birthdays is

$$\frac{365 \cdot 364 \cdots (365-n+1)}{365^n}$$

and the probability of a shared birthday is

$$p_n = 1 - \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}.$$

For $n = 23$, the answer is

$$p_{23} = 0.507.$$

Example: Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Another Example

A couple has two children.

(a) What is the probability that both are girls given that the oldest is a girl?

(b) What is the probability that both are girls given that one of them is a girl?

Let $\Omega = \{(f, f), (f, m), (m, f), (m, m)\}$ (gender of older followed by gender of younger).

Let $A =$ "both are girls" $= \{(f, f)\}$.

Let $B =$ "oldest is a girl" $= \{(f, f), (f, m)\}$.

Let $C =$ "at least one is a girl" $= \{(f, f), (f, m), (m, f)\}$.

(a) $P(A|B) = 1/2$.

(b) $P(A|C) = 1/3$.

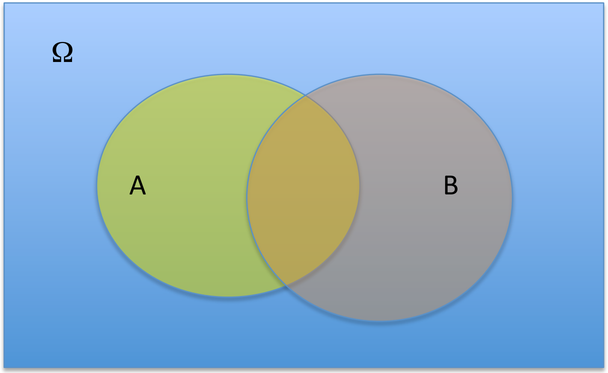
And Another

Show that if A and B are independent, then A and B^c are independent and so are A^c and B^c.

Given: $P(A \cap B) = P(A)P(B)$.

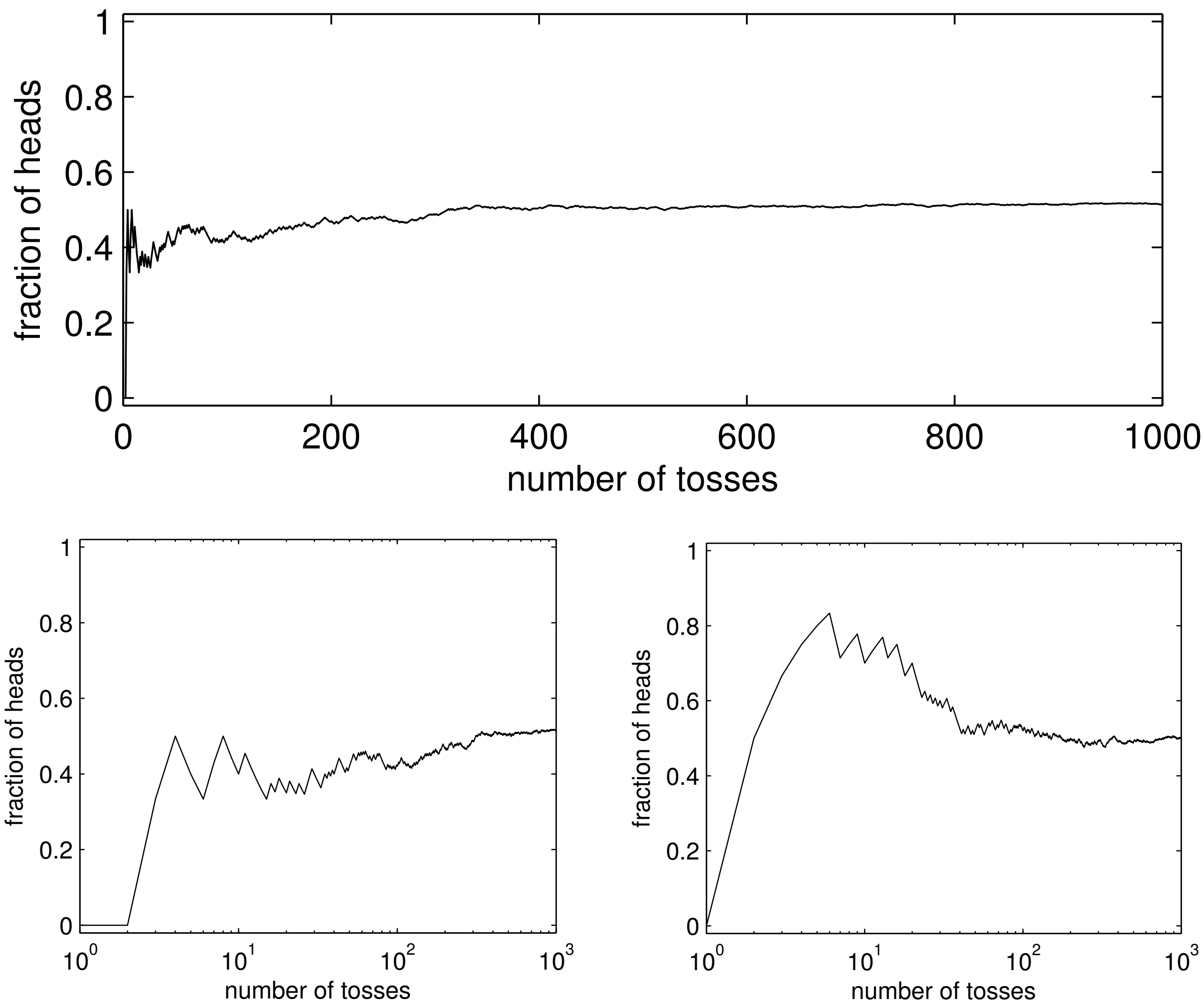
Compute:

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)\left(1 - P(B)\right) \\ &= P(A)P(B^c) \end{aligned}$$



$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= (1 - P(A))\left(1 - P(B)\right) \\ &= P(A^c)P(B^c) \end{aligned}$$

Frequentist View – Coin Tossing



Matlab Code

```
ht = randi([0,1],[1,1000]);
St = cumsum(ht);
Xt = St./(1:1000);

figure(1);
plot(1:1000,Xt,'k-');
ylim([-0.02 1.02]);
xlabel('number of tosses');
ylabel('fraction of heads');
```