Practice Final Exam
Solutions.

(2) $P(X \le 1) = P\left(\frac{X-2}{\sqrt{4}} \le \frac{1-2}{\sqrt{4}}\right) = P(N \le -1/2)$ biggest

$P(Y \le -2) = P\left(\frac{Y-0}{\sqrt{9}} \le \frac{-2-0}{\sqrt{9}}\right) = P(N \le -2/3)$

$P(Z \le 12) = P\left(\frac{Z-15}{\sqrt{16}} \le \frac{12-15}{\sqrt{16}}\right) = P(N \le -3/4)$ smallest

(3) Let $X = G - B$ (good - bad)

$$\bar{X} = \frac{25+30+40-5+35+40+25+30+45+40}{10} = \frac{305}{10} = 30.5$$

$$\overline{X^2} = \frac{625+900+1600+25+1225+1600+625+900+2025+1600}{10}$$

$$= \frac{11175}{10} = 1117.5$$

unbiased $S^2 = \frac{n}{n-1}\overline{(X-\bar{X})^2} = \frac{n}{n-1}\left(\overline{X^2} - \bar{X}^2\right)$

$$= \frac{10}{9}\left(1117.5 - 30.5^2\right) = 14.23$$

$H_0: \mu_X = 0$

$H_1: \mu_X \ne 0$

$$\frac{\bar{X}-0}{S/\sqrt{n}} = \frac{30.5}{14.23/\sqrt{10}} = 6.78$$

$t_{9,\,0.995} = 3.25$ $\Rightarrow$ reject null hypothesis

(4) $F_Y(y) = P(Y \le y) = P\left(e^{\mu+\sigma X} \le y\right) = P\left(\mu + \sigma X \le \ln(y)\right)$

$$= P\left(X \le \frac{\ln(y)-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\frac{\ln(y)-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

(Chain Rule!)

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\ln y - \mu}{\sigma}\right)^2/2} \cdot \frac{1}{\sigma} \cdot \frac{1}{y}$$

(5) (a) $\mathbb{E}\bar{X} = \lambda$

$\mathrm{Var}(\bar{X}) = \lambda/n$

$$\mathbb{P}\left(-z_{\alpha/2} \le \frac{\bar{X}-\lambda}{\sqrt{\lambda/n}} \le z_{\alpha/2}\right) = 1-\alpha$$

Solve the equations $\frac{\bar{X}-\lambda}{\sqrt{\lambda/n}} = \pm z_{\alpha/2}$

$$\Updownarrow$$

$$\bar{X} - \lambda = \pm z_{\alpha/2}\sqrt{\frac{\lambda}{n}}$$

$$\bar{X}^2 - 2\bar{X}\lambda + \lambda^2 = z_{\alpha/2}^2 \frac{\lambda}{n}$$

$$\lambda^2 - \left(2\bar{X} + \frac{z_{\alpha/2}^2}{n}\right)\lambda + \bar{X}^2 = 0$$

$$\lambda = \frac{2\bar{X} + \frac{z_{\alpha/2}^2}{n} \pm \sqrt{\left(2\bar{X} + \frac{z_{\alpha/2}^2}{n}\right)^2 - 4\bar{X}^2}}{2}$$

$$= \bar{X} + \frac{z_{\alpha/2}^2}{2n} \pm \sqrt{\bar{X}\frac{z_{\alpha/2}^2}{n} + \frac{z_{\alpha/2}^4}{4n^2}}$$

$$\lambda = \bar{X} + \frac{z_{\alpha/2}^2}{2n} \pm z_{\alpha/2}\sqrt{\bar{X}/n + \frac{z_{\alpha/2}^2}{4n^2}}$$

Therefore,

$$-z_{\alpha/2} \le \frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} \le z_{\alpha/2}$$

$$\Updownarrow$$

$$\bar{X} + \frac{z_{\alpha/2}^2}{2n} - z_{\alpha/2}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha/2}^2}{4n^2}} \le \lambda \le \bar{X} + \frac{z_{\alpha/2}^2}{2n} + z_{\alpha/2}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha/2}^2}{4n^2}}$$

(b) The "center" of the confidence interval changed from $\bar{X}$ to $\bar{X} + \frac{z_{\alpha/2}^2}{2n}$

The "half-width" of the interval changed from

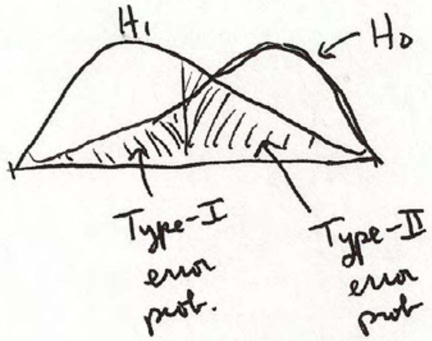
$$z_{\alpha/2}\sqrt{\frac{\bar{X}}{n}} \quad \text{to} \quad z_{\alpha/2}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha/2}^2}{4n^2}}.$$

As n grows larger and larger, the differences get smaller.

(6) (a) H_0: bias favors heads (i.e. $p = 2/3$)

H_1: bias favors tails (i.e. $p = 1/3$)

(b) Here's a graphic representation



If we want Type-I error prob. to equal type-II error prob, then clearly by symmetry the threshold should be at $\boxed{p = 1/2}$

(c) Prob of Type-I error $= P_{H_0}(\bar{X} < 1/2)$

$= P_{H_0}(X_1 + \cdots + X_7 \le 7/2)$

$= p(0) + p(1) + p(2) + p(3)$

$= 0.0005 + 0.0064 + 0.0384 + 0.1280$

$= 0.1733$

(d) By symmetry, Prob of Type-II error $= 0.1733$

(7) 95% conf. interval: $\hat{p} - 1.96\sqrt{\hat{p}\hat{q}/n} \le p \le \hat{p} + 1.96\sqrt{\hat{p}\hat{q}/n}$

$\hat{p} = \frac{60}{100} = 3/5$

$\hat{q} = 1 - \hat{p} = 2/5$

$$\frac{3}{5} - 1.96\underbrace{\sqrt{\frac{3/5 \cdot 2/5}{100}}}_{\frac{\sqrt{6}}{50}} \le p \le \frac{3}{5} + 1.96\sqrt{\frac{3/5 \cdot 2/5}{100}}$$

$$\boxed{0.504 \le p \le 0.696}$$

(8) (a) $\mu = \mathbb{E}X = \int_0^\infty x \frac{1}{\beta} e^{-x/\beta}\,dx = \beta\int_0^\infty y e^{-y}\,dy = \boxed{\beta}$

$y = x/\beta$

$dy = \frac{1}{\beta}dx$

(b) $\sigma^2 = \mathbb{E}\left((X-\mu)^2\right) = \mathbb{E}X^2 - \mu^2 = 2\beta^2 - \beta^2 = \boxed{\beta^2}$

$\mathbb{E}X^2 = \int_0^\infty x^2 \frac{1}{\beta} e^{-x/\beta}\,dx = \beta^2\int_0^\infty y^2 e^{-y}\,dy = 2\beta^2$

(c) Since β is the mean, a good estimator for β would be $\boxed{\overline{X}}$.

(d) $\bar{x} = 0.3336$

unbiased $s^2 = \frac{n}{n-1}\left(\overline{x^2} - \bar{x}^2\right) = \frac{49}{48}\left(0.2812 - 0.3336^2\right)$

$= 0.1735$

$$\bar{x} - 1.96\frac{s}{\sqrt{n}} \le \beta \le \bar{x} + 1.96\frac{s}{\sqrt{n}}$$

$$\boxed{0.2170 \le \beta \le 0.4502}$$

(9) (a) $\hat{\beta}_x = \frac{\overline{tX} - \bar{t}\,\overline{X}}{\overline{t^2} - \bar{t}^2} = \frac{607.8262 - 0.8967 \cdot 679.1160}{1.4116 - 0.8967^2}$

$= \boxed{-1.8717}$

$\hat{\alpha}_x = \overline{X} - \hat{\beta}_x \bar{t} = 679.1160 - (-1.8717)0.8967$

$= \boxed{680.79}$

(b) From the formula for S^2 given on a later page,

$$S^2 = \frac{1}{n-2}\sum_i \left(X_i - \hat{\alpha} - \hat{\beta} t_i\right)^2$$

$$= \frac{n}{n-2}\,\frac{1}{n}\sum_i \left(X_i^2 + \hat{\alpha}^2 + \hat{\beta}^2 t_i^2 - 2\hat{\alpha} X_i - 2\hat{\beta} X_i t_i + 2\hat{\alpha}\hat{\beta} t_i\right)$$

$$= \frac{n}{n-2}\left(\overline{X^2} + \hat{\alpha}^2 + \hat{\beta}^2\,\overline{t^2} - 2\hat{\alpha}\,\overline{X} - 2\hat{\beta}\,\overline{Xt} + 2\hat{\alpha}\hat{\beta}\,\bar{t}\right)$$

$$= \frac{6}{4}\Big(461201 + 463480 + 1.8717^2 \cdot 1.4116 - 2(680.79)(679.116) - 2(-1.8717)(607.8262) + 2(680.79)(-1.8717)\Big)$$

$$S^2 = 7.9592$$

From the t-dist. table, we get $t_{4,0.975} = 2.776$

So, 95% conf. interval is

$$\beta_x = \hat{\beta}_x \pm t_{4,0.975} \frac{S}{\sqrt{n}} \frac{1}{\sqrt{\overline{t^2} - \bar{t}^2}}$$

$$= -1.8717 \pm 2.776 \frac{\sqrt{7.9592}}{\sqrt{6}} \frac{1}{\sqrt{1.4116 - 0.8967^2}}$$

$$= \boxed{-1.8717 \pm 4.1020}$$

(c) $$\hat{\beta}_y = \frac{\overline{tY} - \bar{t}\bar{Y}}{\overline{t^2} - \bar{t}^2} = \frac{-607.0463 - (0.8967)(-664.7352)}{1.4116 - 0.8967^2}$$

$$= \boxed{-18.0703}$$

$$\hat{\alpha}_y = \bar{Y} - \hat{\beta}_y \bar{t} = -664.7352 - (-18.0703)(0.8967)$$

$$= -648.5316$$

$$S^2 = \frac{6}{4}\Big(442072 + (648.5316)^2 + (18.0703)^2 \cdot (1.4116) - 2(-648.5316)(-664.7352)$$
$$-2(-18.0703)(-607.0463) + 2(-648.5316)(-18.0703) \cdot (0.8967)\Big)$$

$$= 1.1002$$

Conf. Interval: $$\beta_y = -18.0703 \pm 2.776 \frac{\sqrt{1.1002}}{\sqrt{6}} \frac{1}{\sqrt{1.4116 - 0.8967^2}}$$

$$= \boxed{-18.073 \pm 1.5251}$$

(d) Proper Motion $= \sqrt{\beta_x^2 + \beta_y^2} \times 0.575$

$$= \sqrt{(1.8717)^2 + (18.0703)^2} \times 0.575$$

$$= \boxed{10.4 \text{ arcsec/yr}}$$

(10)

(a) $\varepsilon_i = z_i - \alpha \sin(2\pi t_i)$

$$f(\alpha) = \sum_i \varepsilon_i^2 = \sum_i \left(z_i - \alpha \sin(2\pi t_i)\right)^2$$

$$\frac{df}{d\alpha} = \sum_i 2\left(z_i - \hat{\alpha}\sin(2\pi t_i)\right)\left(-\sin(2\pi t_i)\right) = 0$$

$$\sum_i z_i \sin(2\pi t_i) = \hat{\alpha}\sum_i \sin^2(2\pi t_i)$$

$$\boxed{\hat{\alpha} = \frac{\sum_i z_i \sin(2\pi t_i)}{\sum_i \sin^2(2\pi t_i)}}$$

(b)

t_i	$\sin(2\pi t_i)$	$\sin^2(2\pi t_i)$	z_i	$z_i \sin(2\pi t_i)$
-0.4940	-0.0377	0.0014	-0.1410	~~-0.0002~~ 0.0053
0.4610	0.2426	0.0589	0.4833	~~0.0284~~ 0.1172
0.7150	-0.9759	0.9524	-0.8375	~~-0.7976~~ 0.8173
1.3070	0.9365	0.8771	1.1157	~~0.9786~~ 1.0449
1.5470	-0.2910	0.0847	-0.2668	~~-0.0226~~ 0.0776
1.8440	-0.8306	0.6899	-0.6741	~~-0.4654~~ 0.5599
		2.6644		~~-0.2984~~ 2.6224

~~$\hat{\alpha} = \frac{-0.2984}{2.6644} = -0.1145$~~

$$\hat{\alpha} = \frac{2.6224}{2.6644} = \boxed{0.9842}$$

(d) $\text{Parallax} = \dfrac{1}{0.575\,\hat{\alpha}} = \boxed{1.767}$

(c) See next page.

(c) Using $\hat{\alpha}$ compute the estimates for the ε_i's:

$$\underline{\hat{\varepsilon}_i = z_i - \hat{\alpha} \sin(2\pi t_i)}$$

-0.1039
0.2445
0.1230
0.1939
0.0196
0.1434

Draw from these six numbers a sample of size six.

For example,

0.2445
0.0196
0.1939 $\Leftarrow$ Call them $\hat{\hat{\varepsilon}}_i$
0.0196
0.1230
0.1230

Generate new z-values: $\hat{\hat{z}}_i = \hat{\alpha} \sin(2\pi t_i) + \hat{\hat{\varepsilon}}_i$

0.2079
0.2584
-0.7686
0.9414
-0.1634
-0.6945

Use these z-values w/ the t-values to compute a new estimate for α. Call it $\hat{\hat{\alpha}}$. Repeat this process thousands of times. Save in a list all of the $\hat{\hat{\alpha}}$'s. The confidence interval for α via bootstrap can be read off from a histogram of the $\hat{\hat{\alpha}}$'s.