Princeton University
Department of Operations Research
and Financial Engineering

ORF 245
Fundamentals of Statistics

Practice Midterm 2

December 4, 2015
10:00 – 10:50 am

Closed book. No computers. Calculators allowed.

You are permitted to use a one-page two-sided cheat sheet.

Return the exam questions and your cheat sheet with your exam booklet.

Note: Some questions count for more than others. Manage your time appropriately.

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(1) *(10 pts.)* Who are you?

(2) *(10 pts.)* Show that the Poisson probabilities $p_0, p_1, \dots$ can be computed recursively by $p_0 = e^{-\lambda}$ and

$$p_k = \frac{\lambda}{k} p_{k-1}, \qquad k = 1, 2, \dots .$$

Use this scheme to find $\mathbb{P}(X \leq 4)$ for $\lambda = 4.5$

(3) *(10 pts.)* Let $f(x) = (1 + \alpha x)/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$ otherwise, where $-1 \leq \alpha \leq 1$. Show that f is a density and find the corresponding cdf. Find the quartiles and the median of the distribution in terms of α.

(4) *(10 pts.)* The **Weibull** cumulative distribution function is

$$F(x) = 1 - e^{-(x/\alpha)^\beta}, \qquad x \geq 0, \quad \alpha > 0, \ \beta > 0.$$

(a) Find the density function.
(b) Suppose that W has a Weibull distribution. Show that $X = (W/\alpha)^\beta$ has an exponential distribution.

(5) *(10 pts.)* Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β, respectively. Find $\mathbb{P}(T_1 > T_2)$.

(6) *(10 pts.)* Find the joint density of $X+Y$ and X/Y, where X and Y are independent exponential random variables with parameter λ. Show that $X+Y$ and X/Y are independent.

(7) *(10 pts.)* Suppose that a queue has n servers and that the length of time to complete a job is an exponential random variable. If a job is at the top of the queue and will be handled by the next available server, what is the distibution of the waiting time until service? What is the distribution of the waiting time until service of the next job in the queue?

(8) *(10 pts.)* A random square has a side length that is a uniform $[0, 1]$ random variable. Find the expected area of the square.

(9) *(10 pts.)* A random rectangle has sides the lengths of which are independent uniform $[0, 1]$ random variables. Find the expected area of the rectangle.

(10) *(10 pts.)* Let X and Y be jointly distributed random variables with correlation ρ_{XY}. Define the *"standardized"* random variables Z_X and Z_Y is the usual way: $Z_X = (X - \mu_X)/\sigma_X$ and $Z_Y = (Y - \mu_Y)/\sigma_Y$. Show that $\mathrm{Cov}(Z_X, Z_Y) = \rho_{XY}$.

(11) *(10 pts.)* The university administration assures a mathematician that he has only 1 chance in $10,000$ of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years, and always rides the elevator up to his office when he first arrives, what is the probability that he will never be trapped? That he will be trapped once? Twice? Assume that

the outcomes on all the days are mutually independent (a dubious assumption in practice).

(12) *(10 pts.)* Let X and Y be jointly continuous random variables. In terms of the joint probability density function $f_{XY}(x, y)$, find an expression for the probability density function of $Z = X - Y$.