Princeton University
Department of Operations Research
and Financial Engineering

ORF 245
Fundamentals of Statistics

Practice Midterm 1

December 4, 2015
10:00 – 10:50 am

Closed book. No computers. Calculators allowed.

You are permitted to use a one-page two-sided cheat sheet.

Return the exam questions and your cheat sheet with your exam booklet.

Note: Some questions count for more than others. Manage your time appropriately.

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(1) *(10 pts.)* What's your name? Who's your daddy?

(2) *(10 pts.)* Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses?

(3) *(10 pts.)* The probability of being dealt a royal straight flush (ace, king, queen, jack, and ten of the same suit) in poker is about 1.3×10^{-8}. Suppose that an avid poker player sees 100 hands a week, 52 weeks a year, for 20 years.
  (a) What is the probability that she is never dealt a royal straight flush?
  (b) What is the probability that she is dealt exactly two royal straight flushes?

(4) *(10 pts.)* Let $F(x) = 1 - e^{-\alpha x^{\beta}}$ for $x \geq 0$, $\alpha > 0$, $\beta > 0$, and $F(x) = 0$ for $x < 0$. Show that F is a cdf and find the corresponding density.

(5) *(10 pts.)* If f and g are densities, show that $\alpha f + (1-\alpha)g$ is a density, where $0 \le \alpha \le 1$.

(6) *(10 pts.)* Let X and Y have the joint density

$$f(x,y) = \frac{6}{7}(x+y)^2, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

(a) By integrating over the appropriate regions, find $\mathbb{P}(X+Y \le 1)$.
(b) Find the marginal densities of X and Y.
(c) Find the conditional density of Y given that $X = x$ (i.e., $f_{Y|X}(y|x)$).

(7) *(10 pts.)* A credit card has n built-in microchips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential random variable with parameter λ, find the density function of the card's lifetime.

(8) *(10 pts.)* Show that if X is a discrete random variable, taking values on the positive integers, then $\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \ge k)$. Use this result to find the expected value of a geometric random variable.

(9) *(10 pts.)* If X and Y are independent random variables with variance σ^2, find $\mathrm{Cov}(X+Y, X-Y)$.

(10) *(10 pts.)* A fair coin is tossed n times and the number of heads, N, is counted. The coin is then tossed N more times. Find the expected total number of heads generated by this process.

(11) *(5 pts.)* A skeptic gives the following argument to show that there must be a flaw in the central limit theorem: "We know that the sum of independent Poisson random variables follows a Poisson distribution with a parameter that is the sum of the parameters of the summands. In particular, if n independent Poisson random variables, each with parameter $1/n$ are summed, the sum has a Poisson distribution with parameter 1. The Central Limit Theorem says that as n approaches infinity, the distribution of the sum tends to a normal distribution, but the Poisson with parameter 1 is not the normal." What do you think of this argument?

(12) *(10 pts.)* Suppose that $X_1, \dots, X_{20}$ are independent random variables with density functions

$$f(x) = 2x, \qquad 0 \le x \le 1.$$

Let $S = X_1 + \cdots + X_{20}$. Describe how one could use the Central Limit Theorem to approximate $\mathbb{P}(S \le 10)$.

(13) *(10 pts.)* Let U and V be a pair of independent random variables uniformly distributed on the interval $[0, 1]$. Let $X = \max(U, V)$ and let $Y = \min(U, V)$.

(a) Find the joint probability density function, $f_{X,Y}$, of X and Y.

(b) Find the marginal probability density functions of X and Y.