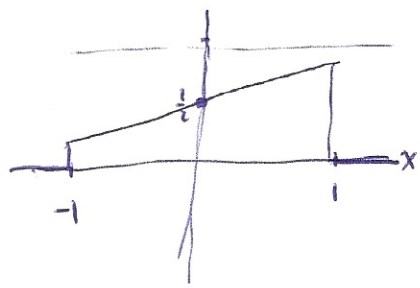
(2) $\quad p_k = \dfrac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2, \ldots$

$\Rightarrow \; p_0 = e^{-\lambda}$

and $\quad p_k = \dfrac{\lambda^k}{k!} e^{-\lambda} = \dfrac{\lambda}{k} \dfrac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \dfrac{\lambda}{k} p_{k-1}$

$\vdots$

(3) $\quad f(x) = \begin{cases} \dfrac{1+\alpha x}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$



f must be nonnegative. Just need to check the endpoints

$0 \le f(-1) = \dfrac{1-\alpha}{2} \quad \Rightarrow \quad \alpha \le 1$ ✓

$0 \le f(1) = \dfrac{1+\alpha}{2} \quad \Rightarrow \quad \alpha \ge -1$ ✓

Area under f must be one:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} \frac{1+\alpha x}{2}\,dx = \underbrace{\int_{-1}^{1} \tfrac{1}{2}\,dx}_{1} + \frac{\alpha}{2} \underbrace{\int_{-1}^{1} x\,dx}_{0}$$ ✓

Cdf: $\quad F(x) = \displaystyle\int_{-\infty}^{x} f(y)\,dy = \int_{-1}^{x} \frac{1+\alpha y}{2}\,dy = \frac{1}{4a}(1+\alpha y)^2 \Big|_{-1}^{x}$

$$= \frac{1}{4a}\left((1+\alpha x)^2 - (1-\alpha)^2\right) \qquad \cancel{0} \le x \le 1$$

Median: $F(x) = 1/2$

$$\overset{\shortparallel}{\frac{1}{4\alpha}\left((1+\alpha x)^2 - (1-\alpha)^2\right)}$$

$$2\alpha = (1+\alpha x)^2 - (1-\alpha)^2$$

$$(1+\alpha x)^2 = 1+\alpha^2$$

$$1+\alpha x = \sqrt{1+\alpha^2}$$

$$x = \frac{\sqrt{1+\alpha^2}-1}{\alpha} \qquad \left(= \frac{\alpha}{\sqrt{1+\alpha^2}+1}\right)$$

Good for all α.

Problematic when $\alpha = 0$

Quartiles are similar

(4) $F(x) = 1 - e^{-(x/\alpha)^\beta}$ $\quad x \geq 0, \quad \alpha > 0, \beta > 0$

(a)
$$f(x) = F'(x) = -e^{-(x/\alpha)^\beta}\left(-\beta (x/\alpha)^{\beta-1} 1/\alpha\right)$$

$$= \frac{\beta x^{\beta-1}}{\alpha^\beta} e^{-(x/\alpha)^\beta} \qquad x > 0$$

(b) $X = \left(\frac{W}{\alpha}\right)^\beta$

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\left(\frac{W}{\alpha}\right)^\beta \leq x\right) = \mathbb{P}\left(\frac{W}{\alpha} \leq x^{1/\beta}\right) = \mathbb{P}(W \leq \alpha x^{1/\beta})$$

$$= 1 - e^{-\left(\frac{\alpha x^{1/\beta}}{\alpha}\right)^\beta} = 1 - e^{-x} \Leftarrow \text{Exponential}$$

(5) $$\mathbb{P}(T_1 > T_2) = \iint_{t_1 > t_2} \alpha e^{-\alpha t_1} \beta e^{-\beta t_2}\, dt_1\, dt_2$$

$$= \int_0^\infty \int_{t_2}^\infty \alpha\beta e^{-\alpha t_1} e^{-\beta t_2}\, dt_1\, dt_2$$

$$= \int_0^\infty \beta e^{-\beta t_2} \underbrace{\int_{t_2}^\infty \alpha e^{-\alpha t_1}\, dt_1}_{e^{-\alpha t_2}}$$

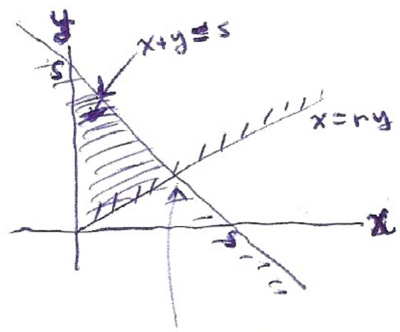
$$= \beta \int_0^\infty e^{-(\alpha+\beta) t_2}\, dt_2$$

$$= \frac{\beta}{\alpha+\beta}$$

(6) Let $S = X + Y$, $R = X/Y$ $\qquad X, Y \sim \text{Exp}(\lambda)$ indep

$$F_{S,R}(s,r) = \mathbb{P}(S \le s, R \le r) = \mathbb{P}(X+Y \le s, X/Y \le r)$$

$$= \mathbb{P}(X+Y \le s, X \le rY)$$



$$= \int_0^{\frac{rs}{1+r}} \int_{x/r}^{s-x} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\, dy\, dx$$

$$= \int_0^{\frac{rs}{1+r}} \lambda e^{-\lambda x} \left[-e^{-\lambda y}\right]_{x/r}^{s-x} dx$$

$x + y = s$, $x = ry$ $\Big\} \Rightarrow (1+r)y = s$

$y = \frac{s}{1+r}$

$x = \frac{rs}{1+r}$

$$= \int_0^{\frac{rs}{1+r}} \lambda e^{-\lambda x}\left(-e^{-\lambda(s-x)} + e^{-\lambda x/r}\right) dx$$

$$= \lambda \int_0^{\frac{rs}{1+r}} \left(-e^{-\lambda s} + e^{-\lambda x - \lambda x/r}\right) dx$$

$$= -\lambda e^{-\lambda s}\frac{rs}{1+r} + \frac{\lambda}{-\lambda(1+1/r)} e^{-\lambda(1+1/r)x}\Big|_0^{\frac{rs}{1+r}}$$

$$= -\lambda e^{-\lambda s}\frac{rs}{1+r} - \frac{r}{1+r}\Big(\underbrace{e^{-\lambda \frac{r+1}{r}\frac{rs}{1+r}}}_{e^{-\lambda s}} - 1\Big)$$

$$= \underbrace{\frac{r}{1+r}}_{F_R(r)}\underbrace{\left(1 - \lambda s e^{-\lambda s} - e^{-\lambda s}\right)}_{F_S(s)}$$

these are cdf's !

Therefore independent !

(7) Let $X_1, X_2, \ldots, X_n$ be the remaining service time of the n servers

The job at the top of the queue will be served at time

$$T = \min(X_1, X_2, \ldots, X_n).$$

Let's compute: $\mathbb{P}(T>t) = \mathbb{P}(X_1>t, \ldots, X_n>t) = P(X>t)^n = (e^{-\lambda t})^n$

$\Rightarrow F_T(t) = 1 - e^{-\lambda n t} \Rightarrow T \sim \text{Exponential}(n\lambda)$

Let's skip the 2nd part!

(8) Let X be the side length and A be the area. $A = X^2$

$$\mathbb{E}(A) = \mathbb{E}(X^2) = \int_0^1 x^2\,dx = \frac{1}{3}$$

(9) $A = XY$

$$\mathbb{E}(A) = \mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

(10)
$$\operatorname{Cov}(Z_X, Z_Y) = \mathbb{E}\left(\frac{X-\mu_X}{\sigma_X}\,\frac{Y-\mu_Y}{\sigma_Y}\right) - \underbrace{\mathbb{E}\left(\frac{X-\mu_X}{\sigma_X}\right)}_{0}\underbrace{\mathbb{E}\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}_{0}$$

$$= \frac{\mathbb{E}\left((X-\mu_X)(Y-\mu_Y)\right)}{\sigma_X\sigma_Y}$$

$$= \frac{\operatorname{Cov}(X,Y)}{\sigma_X\sigma_Y} = \rho_{XY}$$

(11) Number of trappings is Binomial with $p = \frac{1}{10{,}000}$, $n = 5\cdot 52\cdot 10 = 2{,}600$

$$P(X=0) = q^n = \left(1 - \frac{1}{10{,}000}\right)^{2{,}600} = 0.771$$

$$P(X=1) = npq^{n-1} = 2600\,\frac{1}{10000}\,\frac{0.771}{\underbrace{1-\frac{1}{10000}}_{\approx 1}} = 0.200$$

$$P(X=2) = \frac{n(n-1)}{2}p^2q^{n-2} \approx \frac{n^2}{2}p^2q^n = \frac{2600^2}{2}\,\frac{1}{10000^2}\,0.771 = 0.026$$

(12) First find a formula for the cdf:

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(X - Y \le z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y+z} f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{y+z} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dz} \int_{-\infty}^{y+z} f_{XY}(x,y)\, dx \right) dy$$

$$= \boxed{\int_{-\infty}^{\infty} f_{X,Y}(y+z, y)\, dy} \quad \checkmark$$