Practice Midterm

2. 9 heads in 10 tosses is more likely than 18 heads in 20 tosses

Std. dev. in 10 tosses of a fair coin is $\sqrt{npq} = \sqrt{10 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{2.5}$

Std. dev. in 20 tosses of a fair coin is $\sqrt{npq} = \sqrt{20 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{5}$

9 in 10 is 4 above the mean which is $\frac{4}{\sqrt{2.5}}$ std. deviations

18 in 20 is 8 above the mean which is $\frac{8}{\sqrt{5}}$ std. deviations.

$\frac{8}{\sqrt{5}} > \frac{4}{\sqrt{2.5}}$ $\therefore$ 2nd scenario is more std. deviations out.

3. $100 \text{ hands/week} \times 52 \text{ weeks/year} \times 20 \text{ years} = 104000 \text{ hands} \Rightarrow n = 104{,}000$

$p = 1.3 \times 10^{-8}$

$X \sim \text{Binomial}(n, p)$

(a) $P(X=0) = \binom{n}{0} p^0 q^n = q^n = (1-p)^n = (1 - 1.3\times 10^{-8})^{104{,}000} = 0.9986$

(b) $P(X=2) = \binom{n}{2} p^2 q^{n-2} \approx \frac{n^2}{2} p^2 q^n = \frac{104{,}000^2}{2} (1.3\times 10^{-8})^2 (0.9986) = 0.00000914$

4. F is a cdf if $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$, and F is increasing

trivial

true because $\alpha > 0$

will take the derivative and check that it's nonnegative

$$f(x) = F'(x) = \begin{cases} -e^{-\alpha x^\beta}(-\alpha\beta x^{\beta-1}) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

$f(x) \geq 0$ because $\alpha > 0$ and $\beta > 0$

5. f and g are densities which means $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(same for g).

$f \geq 0$ and $g \geq 0$ and $0 \leq \alpha \leq 1 \Rightarrow \alpha f + (1-\alpha) g \geq 0$

$$\int_{-\infty}^{\infty} \left(\alpha f(x) + (1-\alpha) g(x)\right) dx = \alpha \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1} + (1-\alpha) \underbrace{\int_{-\infty}^{\infty} g(x)\,dx}_{1} = \alpha + (1-\alpha) = 1$$

6. X & Y have joint density $f(x,y) = \frac{6}{7}(x+y)^2, \quad 0 \leq x \leq 1, \ 0 \leq y \leq 1.$

(a) $$\mathbb{P}(X+Y \leq 1) = \int_0^1 \int_0^{1-x} \frac{6}{7}(x+y)^2\,dy\,dx = \frac{2}{7}\int_0^1 (x+y)^3 \Big|_{y=0}^{y=1-x} dx$$

$$= \frac{2}{7}\int_0^1 (1^3 - x^3)\,dx = \frac{2}{7}\left(1 - \frac{1}{4}\left[x^4\right]_{x=0}^{x=1}\right)$$

$$= \frac{2}{7}\left(1 - \frac{1}{4}\right) = \frac{2}{7}\frac{3}{4} = \boxed{\frac{3}{14}}$$

(b) $$f_X(x) = \int_0^1 \frac{6}{7}(x+y)^2\,dy = \frac{6}{7}\frac{1}{3}(x+y)^3 \Big|_{y=0}^{y=1} = \frac{2}{7}\left[(x+1)^3 - x^3\right]$$

Similarly, $f_Y(y) = \frac{2}{7}\left[(y+1)^3 - y^3\right]$

(c) $$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}\left[(x+1)^3 - x^3\right]} = 3\,\frac{x^2+2xy+y^2}{3x^2+3x+1}$$

7. Let T = lifetime of card.

$$\mathbb{P}(T > t) = \mathbb{P}(\text{no chip fails by time } t) + \mathbb{P}(\text{one chip fails by time } t)$$

$$= \left(e^{-\lambda t}\right)^n + n\left(e^{-\lambda t}\right)^{n-1}\left(1 - e^{-\lambda t}\right)$$

$$= \left(e^{-\lambda t}\right)^{n-1}\left(e^{-\lambda t} + n - ne^{-\lambda t}\right)$$

$$F_T(t) = 1 - \mathbb{P}(T > t)$$

Chain Rule!

$$f_T(t) = \frac{d}{dt}F_T(t) = -(n-1)\left(e^{-\lambda t}\right)^{n-2} e^{-\lambda t}(-\lambda)\left(e^{-\lambda t} + n - ne^{-\lambda t}\right)$$

$$- \left(e^{-\lambda t}\right)^{n-1}\left(-\lambda e^{-\lambda t} + n\lambda e^{-\lambda t}\right)$$

$$= -\left(e^{-\lambda t}\right)^{n-1}\Big(-\lambda(n-1)e^{-\lambda t} - \lambda(n-1)n + \lambda(n-1)ne^{-\lambda t}$$

$$-\lambda e^{-\lambda t} + n\lambda e^{-\lambda t}\Big)$$

$$= -e^{-(n-1)\lambda t}\left(-\lambda n e^{-\lambda t} - \lambda n(n-1) + \lambda n^2 e^{-\lambda t}\right)$$

$$= -\lambda n e^{-(n-1)\lambda t}\left(1 - e^{-\lambda t} - n\left(1 - e^{-\lambda t}\right)\right)$$

$$= \lambda n(n-1)e^{-(n-1)\lambda t}\left(1 - e^{-\lambda t}\right)$$

8. $$\sum_{k=1}^{\infty}\mathbb{P}(X \geq k) = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\mathbb{P}(X = j) = \sum\sum_{1 \leq k \leq j < \infty}\mathbb{P}(X = j)$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\mathbb{P}(X = j) = \sum_{j=1}^{\infty} j\,\mathbb{P}(X = j) = \sum_{j=0}^{\infty} j\,\mathbb{P}(X = j)$$

$$\| $$

$$\mathbb{E}X$$

(8) continued:

If X is Geometric with parameter p, then

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) = \sum_{k=1}^{\infty} q^{k-1} = \underbrace{\sum_{k=0}^{\infty} q^{k}}_{\text{Geometric series}} = \frac{1}{1-q} = 1/p$$

(9) $$\mathrm{Cov}(X+Y, X-Y) = \mathbb{E}\big((X+Y)(X-Y)\big) - \mathbb{E}(X+Y)\,\mathbb{E}(X-Y)$$

$$= \mathbb{E}(X^2-Y^2) - (\mu_X+\mu_Y)(\mu_X-\mu_Y)$$

$$= \mathbb{E}X^2 - \mathbb{E}Y^2 - (\mu_X^2-\mu_Y^2)$$

$$= (\mathbb{E}X^2-\mu_X^2) - (\mathbb{E}Y^2-\mu_Y^2)$$

$$= \sigma_X^2 - \sigma_Y^2$$

(10) X = total # of heads

$$\mathbb{E}X = \sum_k \mathbb{E}(X \mid N=k)\,\mathbb{P}(N=k) = \sum_k \left(k+\tfrac{1}{2}k\right)\binom{n}{k}\left(\tfrac{1}{2}\right)^n$$

$$= \frac{3}{2}\underbrace{\sum_k k\binom{n}{k}\left(\tfrac{1}{2}\right)^n}_{\text{expected value of } N} = \frac{3}{2}\,\frac{n}{2} = \frac{3}{4}n$$

(11.) The distribution of the X_n's isn't allowed to change as we increase n.

(12) $X_1, X_2, \ldots, X_n$ are iid with $f(x) = 2x$ for $0 \le x \le 1$

$S = X_1 + \cdots + X_{20}$ $\quad$ $n = 20$

$\mu_X = \mathbb{E}X = \int_0^1 x(2x)\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

$\mathbb{E}X^2 = \int_0^1 x^2(2x)\,dx = \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2}$

$\sigma_X^2 = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$

$\mathrm{Var}(S) = n\sigma_X^2$

$\mathbb{P}(S \le 10) = \mathbb{P}(S - n\mu_X \le 10 - n\mu_X)$

$= \mathbb{P}\left(\frac{S - n\mu_X}{\sqrt{n}\,\sigma_X} \le \frac{10 - n\mu_X}{\sqrt{n}\,\sigma_X}\right) = \mathbb{P}(N \le -\sqrt{10})$

Look it up in a normal table

approx. $N(0,1)$

$\frac{10 - 20 \cdot \frac{2}{3}}{\sqrt{20}\sqrt{\frac{1}{18}}} = \frac{-10/3}{\sqrt{10/9}} = -\sqrt{10}$

(13) (a) This part is tricky. Let's skip it for now.

(b) $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(U \le x, V \le x) = \mathbb{P}(U \le x)\,\mathbb{P}(V \le x) = x^2, \quad 0 \le x \le 1$

$f_X(x) = \frac{d}{dx} F_X(x) = 2x, \quad 0 \le x \le 1$

$\mathbb{P}(Y \ge y) = \mathbb{P}(U \ge y, V \ge y) = \mathbb{P}(U \ge y)\,\mathbb{P}(V \ge y) = (1-y)^2$

$F_Y(y) = 1 - (1-y)^2 \quad 0 \le y \le 1$

$f_Y(y) = 2(1-y) \quad 0 \le y \le 1$