ORF 201

COMPUTER METHODS FOR PROBLEM SOLVING

Lecture 20

Shortest Paths: The Last Lab
Decimal vs. Fraction

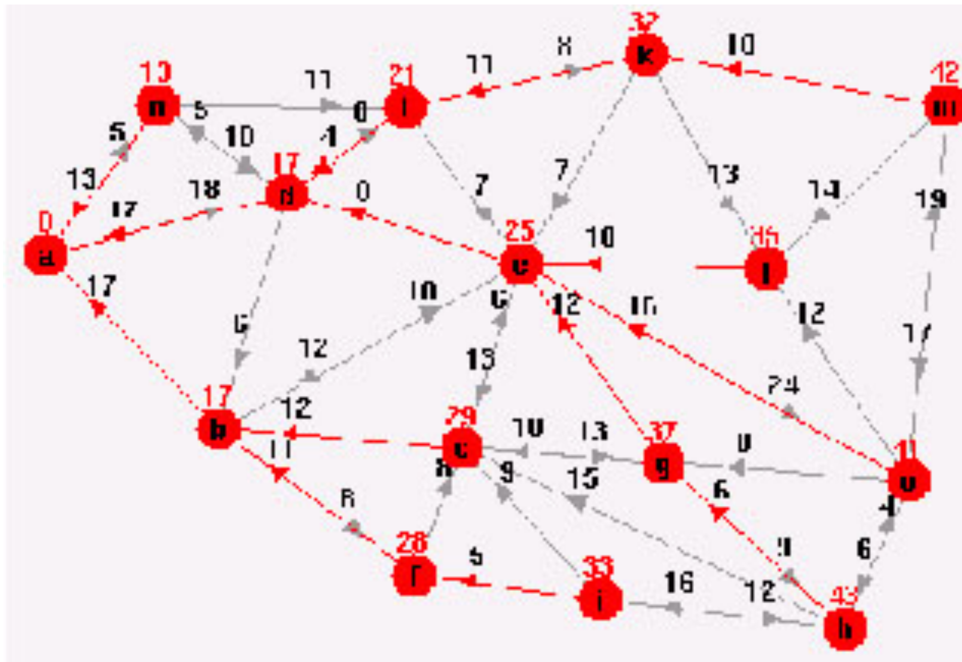
(C) Princeton University

Shortest Paths Problem

Given:

- Network: (N,A)
- Costs = Travel times: c_{ij}, $(i, j) \in A$
- Home (root): $r \in N$

Problem: Find shortest path from every node in N to root.



Dijkstra's Algorithm

Notation:

- Put v_i = min time from i to r
  - Called *label* in networks literature.
  - Called *value* in dynamic programming literature.
- F = set of finished nodes (labels are *set*).
- h_i, $i \in N$ = next node to visit after i (heading).

Dijkstra's Algorithm:

- Initialize:

$$F = \varnothing$$

$$v_j = \begin{cases} 0 & j = r \\ \infty & j \neq r \end{cases}$$

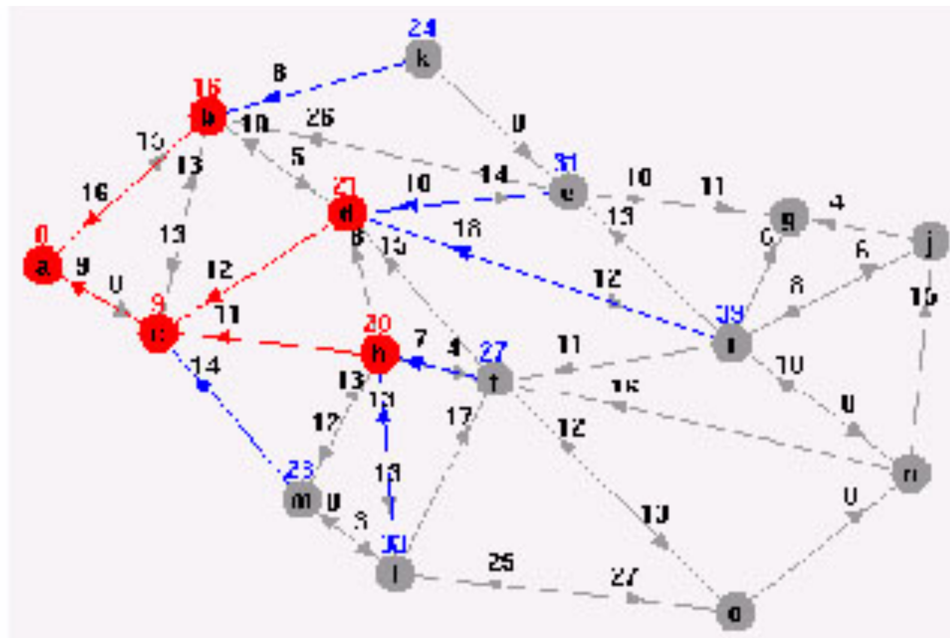
- Iterate:
  - Select unfinished node with smallest v_k. Call it j.
  - Add j to set of finished nodes F.
  - For each unfinished node i having an arc connecting it to j:
    - If $c_{ij} + v_j < v_i$, then set

$$v_i = c_{ij} + v_j$$
$$h_i = j$$

- Stop: when no unfinished nodes remain

Dijkstra's Algorithm - Complexity

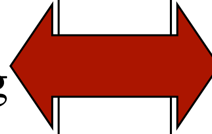
- Each iteration finishes one node: *m* iterations
- Work per iteration:
  - Selecting an unfinished node:
    - Naively, *m* comparisons.
    - Using appropriate data structures, a *heap*, log *m* comparisons.
  - Update adjacent arcs
- Overall: *m* log *m* + *n*



Fractions

Two choices:

Create a **`Fraction`** class consisting of two integers, **`num`** and **`den`**, and write all code using **`Fraction`** instead of double.



Write all code using **`double`** to represent numbers. Convert from/to fraction format on input/output.

Advantages:

`Fraction`

- Seems safer - one implements exactly what one expects: greatest common denominator, reduction to simplest form, etc.

`double`

- Simple to program.
- Very little danger of overflow.
- Easy to switch between decimal and fraction format.

Disadvantages:

- Integer overflow during temporary computation is a danger.
- Code is hard to read: e.g. **`z = x.add(y)`** to add **`Fraction x`** and **`y`** and store in **`z`**.

- Could make small mistakes.

Converting Reals to Fractions

Two methods:

- **Brute Force**: Start with **`den = 1`** and try each possible **`den`** until the associated **`num`** is an integer (with a small tolerance). This works but is terribly inefficient.
- **Continued Fractions**: Represent a real number x by its continued fraction expansion:

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots}}}$$

Truncate after a finite number of terms. This method is amazingly efficient.

Computing the b_j*'s:*

- Put $t_0 = x$
- Put b_j = greatest_integer(t_j)
- Put $t_{j+1} = 1/(t_j - b_j)$.
- Repeat.

Rationalizing Continued Fractions

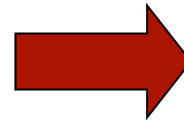
How many terms? Unlike series expansions, you can't just evaluate a continued fraction from left to right stopping when the change gets small.

The obvious way to compute starts with a blind guess of how many terms to use, then starts at the right and works back up to the left.

But there is a beautiful way to compute from left to right.

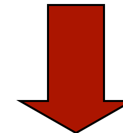
Let:

$$\frac{A_j}{B_j} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ldots + \cfrac{1}{b_j}}}$$



Initialize:

$$A_{-1} = 1 \qquad A_1 = b_0$$
$$B_{-1} = 0 \qquad B_1 = 1$$



Compute each ratio successively:

$$A_j = b_{j-1} A_{j-1} + A_{j-2}$$
$$B_j = b_{j-1} B_{j-1} + B_{j-2}$$

Proof by induction:

First check $j = 1$:

$$\frac{A_1}{B_1} = b_0 + \frac{1}{b_1} = \frac{b_1 b_0 + 1}{b_1} = \frac{b_1 A_0 + A_{-1}}{b_1 B_0 + B_{-1}}$$

Now assume true for all $1, 2, \ldots, j - 1$ and check it for j

Method contFrac in class Format

```
static public String contFrac(int width, int precision, double t)
{
    int bj=0, Aj=0, Aj1, Aj2, Bj=1, Bj1, Bj2, num, den;
    boolean pos;
    double tj, maxDen = Math.pow(10, precision);
    String numstr0, numstr;
    if (t >= 0) { pos = true;  tj =  t; }
    else        { pos = false; tj = -t; }
    bj = (int) (tj+1.0e-12);
    tj = 1/(tj-bj);
    Aj = bj; Aj1 = 1;
    Bj = 1;  Bj1 = 0;
    num = Aj;
    den = Bj;
    if (!pos) {num = -num;}
    numstr0 = "";
    numstr = fracString(num, den, width, precision);
    while (Math.abs(t - Aj/(double)Bj) > 1.0e-12
           && numstr.length() < width && Bj < maxDen ) {
        Aj2 = Aj1; Aj1 = Aj;
        Bj2 = Bj1; Bj1 = Bj;
        bj = (int) (tj+1.0e-12);
        tj = 1/(tj-bj);
        Aj = bj*Aj1 + Aj2;
        Bj = bj*Bj1 + Bj2;
        num = Aj;
        den = Bj;
        if (!pos) {num = -num;}
        numstr0 = numstr;
        numstr = fracString(num, den, width, precision);
    }
    return numstr0;
}
```

A Test Program

```
/***********************************************************
 * Program to compute rational approximations to PI
 **********************************************************/

import myutil.*;

public class ContFrac
{
   public static void main(String[] args)
   {
      double x = Math.PI;

      /****************************************************
       * using static formatting methods                  */

      System.out.println();
      System.out.println("Using static formatting methods");

      System.out.println();
      System.out.prinln("pi = "+Format.floating(10,5,x));
      for (int j=0; j<10; j++) {
         System.out.println("pi = "+Format.contFrac(2*j+1,j,x));
      }
   }
}
```